

# Spans in 2-categories: A monoidal tricategory

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## Abstract

We present Trimble's definition of a tetracategory and prove that the spans in (strict) 2-categories with certain limits have the structure of a monoidal tricategory, defined as a one-object tetracategory. We recall some notions of limits in 2-categories for use in the construction of the monoidal tricategory of spans.

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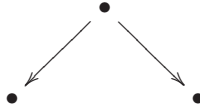
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# 1 Introduction

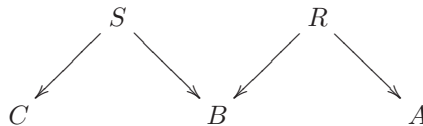
We prove a theorem on the structure of spans in strict 2-categories admitting certain limits. The statement of the theorem is approximately the following:

*Spans in a 2-category with finite limits are the morphisms of a monoidal tricategory.*

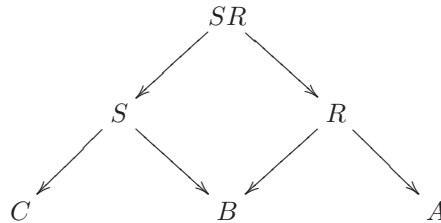
Recall that a ‘span’ in a category  $\mathcal{C}$  is a pair of morphisms with a common domain. This is often drawn in a shape reminiscent of a bridge or roof:



thus spawning a number of different names for this structure, not limited to those listed above. Two spans are composable if they have common codomain and domain, respectively:



and there exists a reasonable notion of pullback, e.g., a limit:



exists. These constructions are variably called ‘pullbacks’, ‘fibered products’, ‘homotopy pullbacks’, ‘weak pullbacks’, ‘pseudo pullbacks’, ‘bipullbacks’, ‘comma objects’, ‘iso-comma objects’, ‘lax pullbacks’, ‘oplax pullbacks’, etc. In some cases, some of these names can be freely interchanged, however some are definitively different from each other. See Kenney and Pronk [16] for a detailed study of composition operations on spans.

This composition operation is *not* strictly associative. This is essentially the original motivation for the generalization from strict 2-categories to bicategories. By introducing a suitable notion of ‘maps of spans’, in 1967, Benabou was able to introduce spans of sets, or, more generally, spans in a category with pullbacks, as an example of a bicategory [5]. In this paper, we categorify Benabou’s work. This requires us to introduce a suitable notion of ‘maps between maps of spans’. These maps are the 3-morphisms of the span construction.

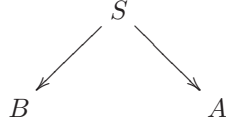
While quite simple in spirit, the span construction is somewhat elusive as it has a tendency to push one towards higher categories. For example, spans in a category are the morphisms of a bicategory, spans in a bicategory are the morphisms of a tricategory, and presumably this pattern continues. However, the construction of a monoidal tricategory of spans presented here is likely to be about as far as one would want to proceed in this explicit fashion without suitable functoriality and coherence results at hand. We define a ‘monoidal tricategory’ to be a one-object tetracategory in the sense of Trimble, following the definition in his 1995 letter to Street [24].

We now say a few words on the structure of this paper. The span construction defined here utilizes particular examples of pseudolimits in 2-categories, which we discuss in an exposition on limits in Section 2. The definition of monoidal tricategory is given in Section 3. The main theorem is the construction of a monoidal tricategory, or one-object tetracategory. This comes in two parts. The first is to construct a tricategory of spans  $\text{Span}(\mathcal{B})$  in Section 4 using pullbacks (by which we mean iso-comma objects). The second is the construction of the monoidal structure on the tricategory of spans in Section 5 using products.

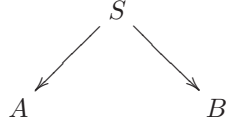
The span construction yields a relatively weak monoidal tricategory, demanding a significant effort in verifying coherence axioms. The techniques used in verifying these axioms all follow very similar reasoning as the components of the structural maps are defined almost invariably by the existence statements of the universal property of pseudolimits, leaving the uniqueness statements as the main tools used in verifying equations. We work through a few of these arguments, but leave most out of the text. Instead we include all of the structural data along with

the equations satisfied by the components of this data. In each instance, this is enough to routinely reproduce and verify the necessary coherence equations.

It is useful in defining and studying the monoidal structure to recall the strong analogy between spans and linear algebra. Spans often appear in constructions of geometric analogues of linear maps and analogues of nice properties of linear maps sometimes hold for such spans. For example, given a span:



one can define a notion of *adjoint* span:



The sense in which this pair of spans are adjoint can be made precise in many settings by associating linear operators to spans in a reasonable way. See, for example, the construction of the *degrouoidification functor* [3].

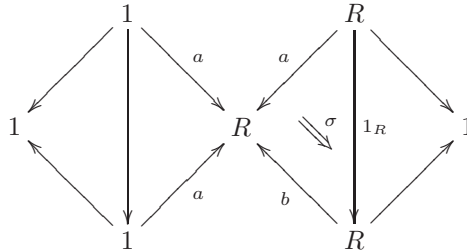
The notion of adjoint span points to a more general notion of ‘internal adjunction’ of morphisms, in this case, within the span construction. In making precise Trimble’s definition of tetracategory we need to give an appropriate notion of equivalence for transformations between tricategories. Defining biadjoint biequivalence structures on each structural transformation provides a notion of equivalence for each level of morphism. A consequence of including this structure is that a notion of ‘adjoint span’ for the structural spans of our construction is built into the monoidal tricategory definition.

## A Note on 2-Cells in Maps of Spans

This paper originated from an interest in monoidal bicategories of spans of groupoids in the context of categorified representation theory in the *groupoidification program* [3, 4]. At the time, we did not have a pressing need for the 3-dimensional structure of spans. Since the examples of interest were bicategories of spans of sets and bicategories of spans of groupoids, it was easy to believe that span bicategories could be constructed from any 2-category with pullbacks.

We can view  $\mathbf{Set}$  as a 2-category where all 2-cells are taken to be identities, and the 2-cells in  $\mathbf{Grpd}$  are natural transformations, whose component 1-cells are morphisms in groupoids, which are, of course, invertible. This commonality turns out to be essential to the construction of spans in a bicategory in which composition is defined by the pullback; specifically, the iso-comma object. It turns out that if the 2-cell components of maps of spans are not required to be *invertible*, then composition may not always be defined. We give a counterexample to illustrate the point.

Consider the 2-category  $\mathbf{Cat}$  of small categories, functors, and natural transformations. Let  $\mathcal{C}$  be the category with objects  $\{A, B\}$ , morphisms  $\{1_A, f: A \rightarrow B, 1_B\}$ , and define functors  $a: R \rightarrow R$  and  $b: R \rightarrow R$ , which send all objects to  $A$  and  $B$ , respectively. Consider the following pair of maps of spans, where  $1$  is the terminal category and  $\sigma: a \rightarrow b1_R$  is the natural transformation that assigns  $f$  to  $A$  and  $1_B$  to  $B$ .



The 2-cells in the pullback are required to be invertible, so they must have only invertible components, i.e., they must be identity natural transformations in this case. Then the composite object on the top is the terminal category and the composite object on the bottom is the initial category. There are no maps from the terminal category to the initial category, so it is not possible to construct a composition functor. Then, to obtain a bicategory or a tricategory  $\mathbf{Span}(\mathbf{Cat})$ , in the sense we desire, the definition of maps of spans should include an invertibility condition on the 2-cells as we have done in Definition 10.

## 2 Limits in 2-Categories

We intend to construct a composition operation by assuming that our 2-category has pullbacks, however, we have not yet been precise about the meaning of pullback, or more generally, the meaning of limit. As suggested in the introduction, we will define composition of spans by pullback of cospan diagrams. There are several closely related limits of cospan diagrams. The construction we work with at present is called the *iso-comma object*, however, we will usually refer to it simply as the *pullback*. Nothing in this section is new, but we hope this exposition will provide a useful introduction to limits in 2-categories.

Limits can be described by *cones*, by *representable functors*, and by *adjoint functors*. We will discuss limits simultaneously as a mixture of the first two. The adjoint functor definition is the most succinct, however we find cones to provide the most intuitive approach and representable functors to be most amenable to generalizations from 1-dimensional limits to weighted and 2-dimensional limits. In this section we will describe conical limits, weighted limits, strict 2-limits, and pseudo limits, looking to ‘pullbacks’ and ‘products’, for examples.

### 2.1 Limits by Cones

While our main interest concerns limits in 2-categories, it will be useful for expository purposes to first recall the definition of limits in 1-categories, as well as examples in the cases of interest.

Intuitively, we think of a limit as the ‘best’ solution to a particular problem. The problem is as follows. Suppose we are given an indexing category  $D$ , which we will call the *shape* of the limit, and a functor  $F: D \rightarrow \mathcal{C}$ , which we will call a *diagram of shape  $D$* . Our first task is to find an object, “ $\text{Lim } F$ ”  $\in \mathcal{C}$ , along with a collection of morphisms  $\{g_d: \text{“Lim } F\text{”} \rightarrow F(d)\}$ , called a *cone*, such that for any morphism  $\delta: d \rightarrow d'$  in  $D$  the triangle

$$\begin{array}{ccc} & \text{“Lim } F\text{”} & \\ g_d \swarrow & & \searrow g_{d'} \\ F(d) & \xrightarrow{F(\delta)} & F(d') \end{array}$$

commutes. Notice the quotes around our potential solution. We mean to signify that this is a possible solution, and when we find the best possible solution, in the sense which we will now describe, then we will remove the quotes and say that we have found *the* limit.

A solution is considered to be the best solution, and thus, the limit, if it satisfies a certain *universal property*. We say a solution  $(\text{Lim } F, \{f_d\})$  is the limit if, for every other possible solution  $(\text{“Lim } F\text{”}, \{g_d\})$ , there exists a unique comparison morphism  $h: \text{“Lim } F\text{”} \rightarrow \text{Lim } F$  in  $\mathcal{C}$  such that for each  $d \in D$ , the triangle

$$\begin{array}{ccc} \text{“Lim } F\text{”} & \xrightarrow{h} & \text{Lim } F \\ g_d \searrow & & \swarrow f_d \\ & F(d) & \end{array}$$

commutes. Note that our use of the definite article ‘the’ is an allowable abuse of language, so long as we remember that the best solution, or limit, is unique only up to the proper notion of equivalence. In a category, this notion of equivalence is *isomorphism*.

It is useful when first encountering limits to do the following exercise: prove that for any two solutions satisfying the universal property stated above, that there is a unique isomorphism between the limit objects. This simple exercise not only gives one a better sense of the universal property, it is also used quite often in limit computations.

### 2.2 Representable functors

We can rephrase the notion of limit in terms of *representable presheaves*. This notion of representability is closely related to the famous Yoneda embedding of a category  $\mathcal{C}$  into its presheaf category:

$$\mathcal{C} \hookrightarrow \text{Set}^{\mathcal{C}}.$$

**Definition.** Let  $\mathcal{C}$  be a locally small category. For each object  $A \in \mathcal{C}$ , there is a contravariant hom-functor

$$\text{hom}(-, A): \mathcal{C}^{\text{op}} \rightarrow \text{Set}.$$

A presheaf (or Set-valued contravariant functor)

$$F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

is said to be **representable** if it is naturally isomorphic to  $\text{hom}(-, A)$  for some object  $A \in \mathcal{C}$ .

Notice that representability is a property of a presheaf. Jumping ahead just for a moment, this property will correspond to the existence of a limit. So, we will say a limit exists when a certain presheaf is representable. However, in practice we often want to work with a specific chosen limit. For this we need the notion of a *representation* of a presheaf.

**Definition.** A **representation** of a presheaf  $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$  is a pair  $(A \in \mathcal{C}, \phi)$ , where

$$\phi: F \Rightarrow \text{hom}(-, A)$$

is a natural isomorphism, i.e., an isomorphism in the functor category  $[\mathcal{C}^{\text{op}}, \text{Set}]$ .

We now want to use this definition to define limits. As our present goal is to understand limits in 2-categories and bicategories, we might want to offer generalized and/or categorified notions of representability, but we find the exposition to be more fluid using only a loose dependence on representations. Having said this, we will not explicitly use the term “representation” in the definition of limit, but its appearance will be very thinly veiled.

## 2.3 Limits of Functors

At the beginning of Section 2.1 we gave an informal definition of limits in terms of families of morphisms called cones. This is probably the most concrete way in which to visualize a limit. In particular, our favorite examples of limits probably provide plenty of intuition for generalizing to a diagram of arbitrary shape. After presenting a more abstract definition of limits by representable functors, we make it clear that this definition matches our definition in terms of cones, and reinforce this equivalence by considering our favorite examples: terminal objects, binary products, and pullbacks.

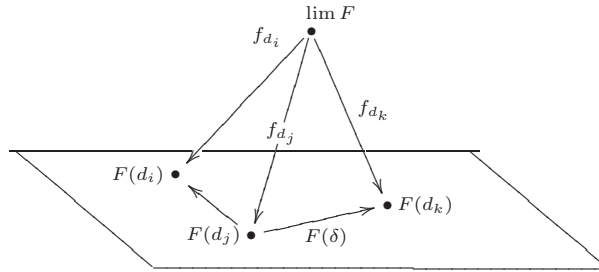
For a fixed shape  $D$  (a category), define  $\Delta 1: D \rightarrow \text{Set}$  to be the *constant point functor*  $d \mapsto \{*\}$ , which takes every object in  $D$  to the singleton set in  $\text{Set}$ .

**Definition.** Let  $\mathcal{C}, D$  be categories and  $F: D \rightarrow \mathcal{C}$  a functor, i.e., a diagram of shape  $D$ . A **limit** of  $F$  is a pair  $(\lim F \in \mathcal{C}, \phi)$ , where

$$\phi: [D, \text{Set}](\Delta 1(-), \mathcal{C}(-, F(-))) \Rightarrow \mathcal{C}(-, \lim F)$$

is a natural isomorphism in  $[\mathcal{C}^{\text{op}}, \text{Set}]$ .

We can now unravel this definition to make the relation to the description by cones explicit. Recall that given a diagram  $F: D \rightarrow \mathcal{C}$ , the limit is an object  $\lim F \in \mathcal{C}$  with a collection of morphisms  $\{f_d \in \mathcal{C}\}_{d \in D}$  called the limiting cone as depicted below.



Unravelling the natural isomorphism  $\phi$  into components, we have, for every object  $A \in \mathcal{C}$ , a *bijection* between sets

$$\phi_A: [D, \text{Set}](\Delta 1(-), \mathcal{C}(A, F(-))) \rightarrow \mathcal{C}(A, \lim F),$$

which sends a natural transformation

$$\sigma: \Delta 1(-) \Rightarrow \mathcal{C}(A, F(-))$$

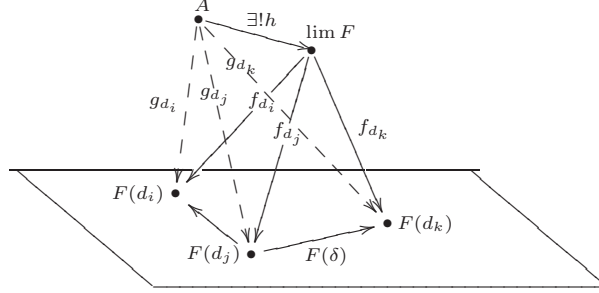
to a morphism

$$\phi_A(\sigma): A \rightarrow \lim F$$

in  $\mathcal{C}$ . We think of the domain of  $\phi_A$  as the set of cones with vertex  $A$ .

To check that the definition of limit just given matches our earlier description of limits, we must

- produce the natural transformation that is the limiting cone with vertex  $\lim F$  defining the limit;
- check that the components of this natural transformation do indeed form a cone by verifying that the component morphisms commute with the morphisms in the diagram;
- and check that for any other cone  $\{g_d\}_{d \in D}$  with vertex object  $A \in \mathcal{C}$ , there is a unique morphism  $h: A \rightarrow \lim F$  such that  $g_d = f_d h$ , for all  $d \in D$ .



We have

- the bijection

$$\phi_{\lim F}^{-1}: \mathcal{C}(\lim F, \lim F) \rightarrow [D, \text{Set}](\Delta 1(-), \mathcal{C}(\lim F, F(-))),$$

which picks out the limiting cone as the image of the identity morphism

$$\phi_{\lim F}^{-1}(1_{\lim F}): \Delta 1(-) \Rightarrow \mathcal{C}(\lim F, F(-)).$$

For each object  $d \in D$ , this natural transformation produces a function

$$\Delta 1(d) = \{*\} \rightarrow \mathcal{C}(\lim F, F(d)),$$

i.e., for each  $d \in D$ , there is a morphism

$$f_d := \phi_{\lim F}^{-1}(1_{\lim F})(d): \lim F \rightarrow F(d)$$

in  $\mathcal{C}$ ;

- the condition that morphisms of the limiting cone  $\{f_d\}_{d \in D}$  commute with the morphisms  $F(\delta)$  in the diagram  $F(D)$ , for each  $\delta: d \rightarrow d'$  in  $D$ , is satisfied by naturality of  $\phi_{\lim F}^{-1}(1_{\lim F})$  from which we obtain the commutative diagram

$$\begin{array}{ccc} & \{*\} & \\ \Delta 1(d) \swarrow & & \searrow \Delta 1(d') \\ \mathcal{C}(\lim F, F(d)) & \xrightarrow{\mathcal{C}(\lim F, F(\delta))} & \mathcal{C}(\lim F, F(d')) \end{array}$$

or unravelled a bit more,

$$\begin{array}{ccc} & \lim F & \\ f_d \swarrow & & \searrow f_{d'} \\ F(d) & \xrightarrow{F(\delta)} & F(d') \end{array}$$

$$F(\delta)f_d = f_{d'};$$

- for an object  $A \in \mathcal{C}$ , the bijection

$$\phi_A: [D, \text{Set}](\Delta 1(-), \mathcal{C}(A, F(-))) \rightarrow \mathcal{C}(A, \lim F),$$

takes a cone

$$g: \Delta 1(-) \Rightarrow \mathcal{C}(A, F(-))$$

to the unique morphism

$$h := \phi_A(\sigma): A \rightarrow \lim F,$$

such that  $g_d = f_d h$  for all  $d \in D$ . This is checked by considering the naturality square for the isomorphism  $\phi$

$$\begin{array}{ccc} \mathcal{C}(\lim F, \lim F) & \xrightarrow{-\circ h} & \mathcal{C}(A, \lim F) \\ \phi_{\lim F} \downarrow & & \downarrow \phi_A \\ [D, \text{Set}](\Delta 1(-), \mathcal{C}(\lim F, F(-))) & \xrightarrow{-\circ h} & [D, \text{Set}](\Delta 1(-), \mathcal{C}(A, F(-))) \end{array}$$

Chasing the identity  $1_{\lim F}: \lim F \rightarrow \lim F$  around the diagram produces the desired result.

Although, we will return to this notion, it is worth mentioning at this point that limits can be vastly generalized from the ‘conical limits’ we consider above, to ‘weighted limits’ in enriched categories. We will introduce weighted limits to obtain our desired notion of iso-comma object, but also to help dispel the common feeling of the uninitiated of being lost in a sea of limits. Note that we will use the notation  $\lim(\Delta 1, F)$  in place of  $\lim F$  in general. In particular, this allows us to replace the ‘conical weight’  $\Delta 1$  with a more general ‘weighting functor’  $W$ , in which case, we write  $\lim(W, F)$ .

## 2.4 Some Finite Limits in 1-Categories

In this section we look at the terminal object, the product and the pullback, each of which is an example of a limit construction. We will define limits in an arbitrary category  $\mathcal{C}$ , but we are not concerned with issues of existence. Since the category of sets is complete, i.e., has all limits, it may be useful to think of our examples as limits in  $\text{Set}$ .

### Terminal Objects

Let  $\mathcal{C}$  be a category and  $D$  the initial object in  $\text{Cat}$ . For a diagram  $F: D \rightarrow \mathcal{C}$ , the limit is an object  $\mathbf{1} \in \mathcal{C}$  with a natural isomorphism

$$\phi: [D, \text{Set}](\Delta 1(-), \mathcal{C}(-, F(-))) \rightarrow \mathcal{C}(-, \mathbf{1}).$$

For every object  $A \in \mathcal{C}$ , there is a bijection

$$\phi_A: [D, \text{Set}](\Delta 1(-), \mathcal{C}(A, F(-))) \rightarrow \mathcal{C}(A, \mathbf{1}),$$

where the domain has exactly one element (since  $D$  has no objects), which gets mapped to a unique morphism  $A \rightarrow \mathbf{1}$  in  $\mathcal{C}$ .

### Products

Let  $\mathcal{C}$  be a category and  $D$  the category with exactly two objects and only identity morphisms. For a diagram  $F: D \rightarrow \mathcal{C}$ , the limit is an object  $\lim F \in \mathcal{C}$  with a natural isomorphism

$$\phi: [D, \text{Set}](\Delta 1(-), \mathcal{C}(-, F(-))) \rightarrow \mathcal{C}(-, \lim F).$$

Note that, in this case, a diagram  $F(D)$  is a pair of objects  $X, Y \in \mathcal{C}$ , so the limiting object  $\lim F$  would, in practice, be written as  $X \times Y$ .

For every object  $A \in \mathcal{C}$ , there is a bijection

$$\phi_A: [D, \text{Set}](\Delta 1(-), \mathcal{C}(A, F(-))) \rightarrow \mathcal{C}(A, \lim F).$$

The pair of morphisms in the limiting cone

$$\phi_{\lim F}^{-1}(1_{\lim F}): \Delta 1(-) \Rightarrow \mathcal{C}(\lim F, F(-))$$

are the “projections”

$$Y \leftarrow \lim F \rightarrow X.$$

The universal property is straightforward to verify. For each object  $A \in \mathcal{C}$ , a cone

$$\sigma: \Delta 1(-) \rightarrow \mathcal{C}(A, F(-)),$$

which is a pair of morphisms

$$Y \leftarrow A \rightarrow X.$$

picks out a unique comparison map

$$\phi_A(\sigma): A \rightarrow \lim F,$$

and

$$\begin{array}{ccc} & A & \\ \swarrow & \phi_A(\sigma) & \searrow \\ Y & \lim F & X \end{array}$$

commutes by naturality of  $\phi$ .

### Pullbacks

Let  $\mathcal{C}$  be a category and  $D$  the category with exactly two non-identity arrows each of which has the same codomain but different domain. The diagrams of this category are the *cospans* in  $\mathcal{C}$ , as pictured below. For a diagram  $F: D \rightarrow \mathcal{C}$ , the limit is an object  $\lim F \in \mathcal{C}$  with a natural isomorphism

$$\phi: [D, \text{Set}](\Delta 1(-), \mathcal{C}(-, F(-))) \rightarrow \mathcal{C}(-, \lim F).$$

As we mentioned, the diagram  $F(D)$  is the familiar cospan

$$\begin{array}{ccc} Y & & X \\ & \searrow & \swarrow \\ & Z & \end{array}$$

in  $\mathcal{C}$ , so the limiting object  $\lim F$ , usually called the *pullback* or *fibered product*, would, in practice, be written as  $X \times_Z Y$ .

For every object  $A \in \mathcal{C}$ , there is a bijection

$$\phi_A: [D, \text{Set}](\Delta 1(-), \mathcal{C}(A, F(-))) \rightarrow \mathcal{C}(A, \lim F).$$

The triple of morphisms in the limiting cone

$$\phi_{\lim F}^{-1}(1_{\lim F}): \Delta 1(-) \Rightarrow \mathcal{C}(\lim F, F(-))$$

are the “projections”

$$\begin{array}{ccc} & \lim F & \\ \swarrow & \downarrow & \searrow \\ Y & & X \\ \searrow & & \swarrow \\ & Z & \end{array}$$

The universal property is straightforward to verify. For each object  $A \in \mathcal{C}$ , a cone

$$\sigma: \Delta 1(-) \rightarrow \mathcal{C}(A, F(-)),$$

which is a triple of morphisms

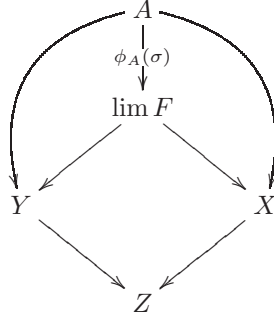
$$\begin{array}{ccc} & A & \\ \swarrow & \downarrow & \searrow \\ Y & & X \\ \searrow & & \swarrow \\ & Z & \end{array}$$



such that the triangles commute, picks out a unique comparison map

$$\phi_A(\sigma): A \rightarrow \lim F,$$

and



commutes by naturality of  $\phi$ .

Note that the projection to  $Z$  in any cone is, in a sense, superfluous, since the triangles formed by the cone and diagram commute. In practice, this morphism is often left out of the definition of limit “by cones”. For pullbacks in 2-categories a similar situation occurs, but requires slightly more care, and leads to the differentiation between *pseudo pullbacks* and *iso-comma objects*.

## 2.5 Strict and Pseudo Limits in 2-Categories

In this section, we are interested in limits in 2-categories.

To define limits in 1-categories, we considered *natural transformations* in the functor category  $[D, \text{Set}]$  as cones. Natural transformations are the 2-morphisms in the 2-category of categories, functors, and natural transformations. In this section we need to generalize our cones to *strict transformations* and *strong transformations*, which are the 2-morphisms in the 3-categories of 2-categories, homomorphisms, (strict or strong) transformations, and modifications, respectively. This distinction between strict and strong transformations is the primary difference between strict and pseudo limits. However, pseudo limits can be imitated by strict limits using weighted limits. We will demonstrate this in our definition of iso-comma object.

Let  $D$  be a (strict) 2-category, then  $[D, \text{Cat}]$  is the 2-category of (strict) 2-functors, strict transformations, and modifications. We denote by  $2\text{Cat}(D, \text{Cat})$  the strict 2-category (which is strict since the codomain is a strict 2-category) of homomorphisms, strong transformations, and modifications. Then  $\text{Ps}(D, \text{Cat})$  will denote the full sub-2-category of (strict) 2-functors in  $2\text{Cat}(D, \text{Cat})$ .

All 2-categorical limits will be considered in the generalized context of weighted limits. At this point, we replace the constant point functor  $\Delta 1: D \rightarrow \text{Set}$  with an arbitrary (strict) 2-functor  $W: D \rightarrow \text{Cat}$ , called the *weight*, and write  $\lim(W, F)$  in place of  $\lim F$ . However, many important examples are still conical limits, i.e., limits weighted by the strict 2-functor  $\Delta 1: D \rightarrow \text{Cat}$  that assigns each object of  $D$  the terminal category and all 1- and 2-morphisms to the respective identity morphisms. Note that weighted limits do show up in the context of 1-categorical limits when extending to limits in enriched category theory. Of course, strict 2-categories are  $\text{Cat}$ -enriched categories, hence the  $\text{Cat}$ -valued weights in the following definitions.

Strict 2-limits and pseudo limits have a common property as 2-categorical limits, which differentiates them from the more general 2-dimensional limits. That is, the universal properties are stated as *isomorphisms* of categories, rather than *equivalences*. Disregarding some of the nuances suggested above, which arise when considering weighted limits, the main difference between strict and pseudo limits is that for strict limits each cone commutes with the diagram “on the nose”, whereas for a pseudo limit, commutativity holds only up to specified isomorphisms. This distinction arises by replacing  $[D, \text{Cat}]$  in the definition of strict limits with  $\text{Ps}(D, \text{Cat})$  in the definition of pseudo limits.

We now recall the definitions.

**Definition.** Let  $\mathcal{C}, D$  be (strict) 2-categories,  $F: D \rightarrow \mathcal{C}$  a diagram, and  $W: D \rightarrow \text{Cat}$  a weight, both (strict) 2-functors. A **strict  $W$ -weighted 2-limit** of  $F$  is a pair  $(\lim(W, F) \in \mathcal{C}, \phi)$ , where

$$\phi: [D, \text{Cat}](W(-), \mathcal{C}(-, F(-))) \Rightarrow \mathcal{C}(-, \lim(W, F))$$

is an invertible strict transformation in  $[\mathcal{C}, \text{Cat}]$ .

**Definition.** Let  $\mathcal{C}, D$  be (strict) 2-categories,  $F: D \rightarrow \mathcal{C}$  a diagram, and  $W: D \rightarrow \mathbf{Cat}$  a weight, both (strict) 2-functors. A  **$W$ -weighted pseudo limit** of  $F$  is a pair  $(\text{pslim}(W, F) \in \mathcal{C}, \phi)$ , where

$$\phi: \mathbf{Ps}(D, \mathbf{Cat})(W(-), \mathcal{C}(-, F(-))) \Rightarrow \mathcal{C}(-, \text{pslim}(W, F))$$

is an invertible strict transformation in  $[\mathcal{C}, \mathbf{Cat}]$ .

The components of  $\phi$  are isomorphisms of categories in each of the preceding definitions.

## 2.6 Some Finite Limits in 2-Categories

### Terminal Objects

Let  $\mathcal{C}$  be a (strict) 2-category and  $D$  be the initial 2-category, i.e., the 2-category that has an empty set of objects. For the unique diagram  $F: D \rightarrow \mathcal{C}$ , the limit is an object  $\mathbf{1} \in \mathcal{C}$  with an invertible strict transformation

$$\phi: [D, \mathbf{Cat}](\Delta \mathbf{1}(-), \mathcal{C}(-, F(-))) \rightarrow \mathcal{C}(-, \mathbf{1}).$$

For every object  $A \in \mathcal{C}$ , there is an isomorphism of categories

$$\phi_A: [D, \mathbf{Cat}](\Delta \mathbf{1}(-), \mathcal{C}(A, F(-))) \rightarrow \mathcal{C}(A, \mathbf{1}),$$

where the domain has exactly one transformation (since  $D$  has no objects), which gets mapped to a unique morphism  $A \rightarrow \mathbf{1}$  in  $\mathcal{C}$ .

It is not difficult to check that strict terminal objects as we have defined them are also pseudo limits.

### Products

Let  $\mathcal{C}$  be a (strict) 2-category and  $D$  the (strict) 2-category with exactly two objects and only identity 1- and 2-morphisms. For a diagram  $F: D \rightarrow \mathcal{C}$ , the limit is an object  $\lim F \in \mathcal{C}$  with an invertible strict transformation

$$\phi: [D, \mathbf{Cat}](\Delta \mathbf{1}(-), \mathcal{C}(-, F(-))) \rightarrow \mathcal{C}(-, \lim F).$$

Note that, in this case, a diagram  $F(D)$  is a pair of objects  $X, Y \in \mathcal{C}$  with just identity 1- and 2-morphisms, so the limiting object  $\lim F$  would, in practice, be written as  $X \times Y$ .

For every object  $A \in \mathcal{C}$ , there is an isomorphism of categories

$$\phi_A: [D, \mathbf{Cat}](\Delta \mathbf{1}(-), \mathcal{C}(A, F(-))) \rightarrow \mathcal{C}(A, \lim F).$$

The pair of 1-morphisms in the limiting cone

$$\phi_{\lim F}^{-1}(\mathbf{1}_{\lim F}): \Delta \mathbf{1}(-) \Rightarrow \mathcal{C}(\lim F, F(-))$$

are the “projections”

$$Y \leftarrow \lim F \rightarrow X.$$

The universal property is straightforward to verify.

- Since the components of  $\phi$  are isomorphisms, for each object  $A \in \mathcal{C}$ , a cone

$$\sigma: \Delta \mathbf{1}(-) \rightarrow \mathcal{C}(A, F(-)),$$

which consists of a pair of 1-morphisms

$$Y \leftarrow A \rightarrow X.$$

is mapped to a unique comparison map

$$\phi_A(\sigma): A \rightarrow \lim F,$$

and we see that

$$\begin{array}{ccc} & A & \\ \swarrow & \phi_A(\sigma) & \searrow \\ Y & \xleftarrow{\quad} \lim F \xrightarrow{\quad} & X \end{array}$$

commutes by naturality of  $\phi$ . This is the one-dimensional aspect of the universal property.

- Now, consider an object  $A \in \mathcal{C}$  and a pair of 1-morphisms  $h, k: A \rightarrow \lim F$  such that for each  $d \in D$ , there is a 2-morphism

$$\phi_{\lim F}^{-1}(1_{\lim F})_d h \Rightarrow \phi_{\lim F}^{-1}(1_{\lim F})_d k.$$

Since there are no non-trivial morphisms in  $D$ , it is immediate that this collection of 2-morphisms is a modification in  $[D, \text{Cat}](\Delta 1(-), \mathcal{C}(A, F(-)))$ , i.e., a map of cones. Since  $\phi_A$  is an isomorphism of categories, and thus, fully faithful, this modification maps to a unique 2-morphism  $h \Rightarrow k$  in  $\mathcal{C}$ . This is the two-dimensional aspect of the universal property.

As with terminal objects, strict products as we have defined them are also pseudo limits.

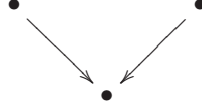
## Pullbacks

In 2-categories, pullbacks are more interesting as limits than terminal objects and products. The conical strict 2-limit with the usual diagram for pullbacks is not a pseudo limit. This is closely related to the fact that this limit requires diagrams of 1-morphisms to commute on the nose, which is generally not a desirable property in a 2-category. We usually ask for a diagram to commute up to a 2-morphism, so we work instead with the pseudo pullback.

We can define the conical strict pullback, which is a representation of the 2-functor

$$[D, \text{Set}](\Delta 1(-), \mathcal{C}(-, F(-))): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$$

where  $D$  is the 2-category



with no non-identity 2-morphisms. Unravelling this definition gives the pullback as a 1-categorical limit, which we defined previously, but with a 2-categorical universal property.

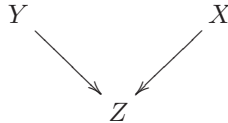
Since we will not use the strict pullback, we will not give anymore details. Instead, we will define the pseudo pullback as a conical pseudo limit. This is a good opportunity to give some intuition for non-conical limits as well. Any conical pseudo limit can be defined as a non-conical strict 2-limit. We will first define the pseudo pullback as a conical, or  $\Delta 1$ -weighted, pseudo limit.

Let  $\mathcal{C}$  be a (strict) 2-category and  $D$  as above. For a (strict) 2-functor, or diagram,  $F: D \rightarrow \mathcal{C}$ , the pseudo limit is an object  $\lim F \in \mathcal{C}$  with an invertible strict transformation

$$\phi: \text{Ps}(D, \text{Cat})(\Delta 1(-), \mathcal{C}(-, F(-))) \rightarrow \mathcal{C}(-, \lim F)$$

in  $[\mathcal{C}, \text{Cat}]$ .

Recall, a diagram  $F(D)$  is a cospan



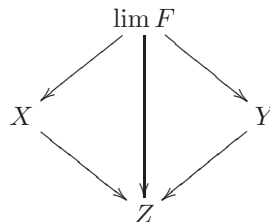
in  $\mathcal{C}$ , so again the limiting object  $\lim F$  would, in practice, be written as  $X \times_Z Y$ . For every object  $A \in \mathcal{C}$ , there is an isomorphism of categories

$$\phi_A: \text{Ps}(D, \text{Cat})(\Delta 1(-), \mathcal{C}(A, F(-))) \rightarrow \mathcal{C}(A, \lim F).$$

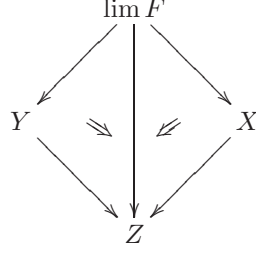
There are three morphisms in the limiting cone

$$\phi_{\lim F}^{-1}(1_{\lim F}): \Delta 1(-) \Rightarrow \mathcal{C}(\lim F, F(-))$$

coming from the three objects of  $D$ . These are the “projections”



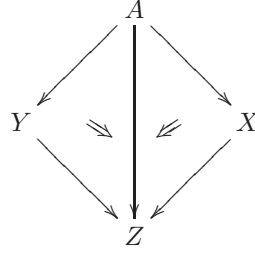
Since the cone is now a strong transformation rather than the strict transformations defining cones for strict 2-limits, each 1-morphism in  $D$  gives us an invertible 2-morphism in  $\mathcal{C}$ . It follows that the pseudo pullback has 2-cells as pictured:



- The universal property is straightforward to verify. For each object  $A \in \mathcal{C}$ , an object

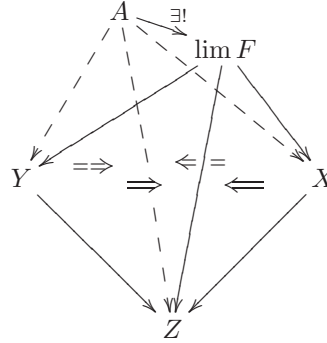
$$\sigma: \Delta 1(-) \rightarrow \mathcal{C}(A, F(-)),$$

in  $\text{Ps}(D, \text{Cat})(\Delta 1(-), \mathcal{C}(A, F(-)))$  is a cone



We obtain a unique comparison map

$$\phi_A(\sigma): A \rightarrow \lim F,$$



commutes by naturality of  $\phi$ . That is, since  $\phi$  is a strict invertible transformation, the cone at vertex  $A$  must be equal to the limiting cone precomposed with the 1-morphism  $\phi_A(\sigma)$ . This amounts to a pair of 1-cell equations and a pair of 2-cell equations, which appear on the left and right sides of the above diagram.

- Now, consider an object  $A \in \mathcal{C}$  and a pair of 1-morphisms  $h, k: A \rightarrow \lim F$  such that for each  $d \in D$ , there is a 2-morphism

$$M_d: \phi_{\lim F}^{-1}(1_{\lim F})_d h \Rightarrow \phi_{\lim F}^{-1}(1_{\lim F})_d k$$

satisfying for each  $\delta: d \rightarrow d'$  in  $D$ , an equation

$$\phi_{\lim F}^{-1}(1_{\lim F})(\delta) \circ F(\delta) \cdot M_d = M_{d'} \circ \phi_{\lim F}^{-1}(1_{\lim F})(\delta)$$

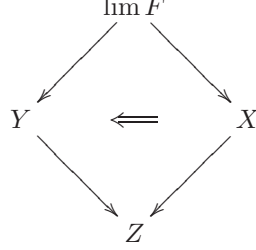
between 2-morphisms in  $\mathcal{C}$ . These are the equations expressing that the collection of 2-morphisms  $\{M_d\}_{d \in D}$  is a modification, i.e., a 1-morphism in  $\text{Ps}(D, \text{Cat})(\Delta 1(-), \mathcal{C}(A, F(-)))$ . Since  $\phi_A$  is an isomorphism of categories, and thus, fully faithful, this modification maps to a unique 2-morphism  $\gamma: h \Rightarrow k$  in  $\mathcal{C}$  such that for each  $d \in D$

$$\phi_{\lim F}^{-1}(1_{\lim F})_d \cdot \gamma = M_d.$$

This is the two-dimensional aspect of the universal property.

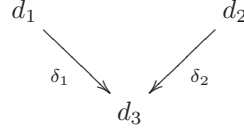
In the definition that follows, we do not require the modification axioms as an extra condition. We circumvent this requirement by defining the pullback as a weighted strict 2-limit rather than a pseudo 2-limit. The analogous condition to the equation relating  $M_d$  and  $M_{d'}$  above, will be the naturality equation, which we will see is non-trivial in weighted limits.

Since the 2-cells of the cones are invertible, one often inverts one of the 2-cells, composes the resulting pair, and then discards the projection to  $Z$  when defining the pseudo pullback by cones. The resulting limit is called the *iso-comma object*.



As suggested at the beginning of this section, pseudo pullbacks can also be defined as non-conical strict 2-limits. This definition is actually closer to that of the iso-comma object, since there are only 2 “distinct” projection maps and one invertible 2-morphism.

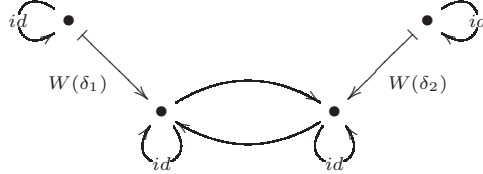
We take the same definition for the 2-category  $D$ :



but now define a non-conical weighting 2-functor  $W : D \rightarrow \text{Cat}$ . The idea is that while the cones for strict 2-limits do not contain 2-morphisms, this weakened form of commutativity can be introduced by a non-conical weighting functor. Let  $\mathbf{1}$  be the terminal 2-category. We make the assignments

$$\begin{aligned} d_1 &\mapsto W(d_1) := \text{a point with a self-loop labeled } id \\ d_2 &\mapsto W(d_2) := \text{a point with a self-loop labeled } id \\ d_3 &\mapsto W(d_3) := \text{two points, each with a self-loop labeled } id, \text{ connected by an invertible pair of arrows} \end{aligned}$$

where the two non-identity arrows are an invertible pair. Then the image of  $W$  in  $\text{Cat}$  is a pair of functors mapping the terminal categories  $W(d_1)$  and  $W(d_2)$  each to one of the pair of objects in  $W(d_3)$



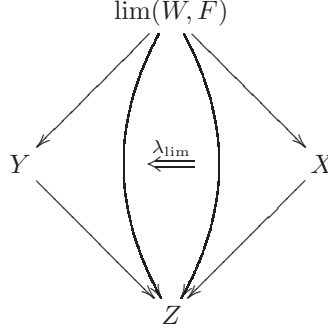
For a strict 2-functor  $F : D \rightarrow \mathcal{C}$ , the  $W$ -weighted strict 2-pullback of shape  $D$  is an object  $\lim(W, F)$  together with a strict invertible transformation

$$\phi : [D, \text{Cat}](W(-), \mathcal{C}(-, F(-))) \Rightarrow \mathcal{C}(-, \lim(W, F))$$

in  $[\mathcal{C}, \text{Cat}]$ . The strict transformation

$$\phi_{\lim(W, F)}^{-1}(1_{\lim(W, F)}) : W(-) \rightarrow \mathcal{C}(\lim(W, F), F(-)).$$

is the limiting cone



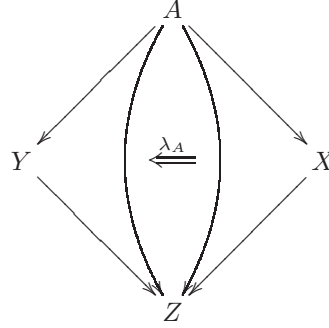
The morphisms to  $X$  and  $Y$  come from the single object of  $W(d_1)$  and  $W(d_2)$ , respectively. Due to the non-conical weighting  $W$ , there are two morphisms in the cone with codomain  $Z$ , each coming from one of the objects in  $W(d_3)$ . The invertible pair of morphisms in  $W(d_3)$  yield the invertible 2-morphism in the diagram and thereby “weaken” the usual strict pullback.

The universal property is as follows.

- For each object  $A \in \mathcal{C}$ , an object

$$\sigma: W(-) \rightarrow \mathcal{C}(A, F(-)),$$

in  $[D, \text{Cat}](W(-), \mathcal{C}(A, F(-)))$  is a cone



Since  $\phi_A$  is an isomorphism, we obtain a unique comparison map

$$\phi_A(\sigma): A \rightarrow \lim F,$$

such that

$$\phi_{\lim(W, F)}^{-1}(1_{\lim(W, F)})_d \phi_A(\sigma) = \sigma_d$$

and

$$\lambda_{\lim} \cdot \phi_A(\sigma) = \lambda_A.$$

This follows from  $\phi$  being a strict invertible transformation. This is the one-dimensional aspect of the universal property.

- Now, consider an object  $A \in \mathcal{C}$  and a pair of 1-morphisms  $h, k: A \rightarrow \lim(W, F)$  such that for each object  $d \in D$ , there is a natural transformation

$$M_d: \phi_{\lim(W, F)}^{-1}(1_{\lim(W, F)})_d h \Rightarrow \phi_{\lim(W, F)}^{-1}(1_{\lim(W, F)})_d k.$$

The naturality equation is analogous to the modification equation required in our previous discussion of the pseudo pullback.

The maps  $\{M_d \mid d \in D\}$  are the components of a modification. Since  $\phi_A$  is an isomorphism of categories, and thus, fully faithful, this modification maps to a unique 2-morphism  $\gamma: h \Rightarrow k$  in  $\mathcal{C}$  such that for each  $d \in D$ ,

$$\phi_{\lim(W, F)}^{-1}(1_{\lim(W, F)})_d \cdot \gamma = M_d.$$

This is the two-dimensional aspect of the universal property.

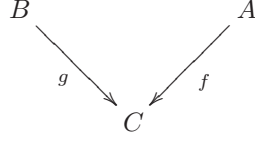
In fact, this is equivalent to the definition we will work with throughout this paper and which we will refer to as the pullback. See Section 2.7 for a more explicit definition of the pullback and its universal property.

## 2.7 Explicit Definitions of Pullbacks and Products

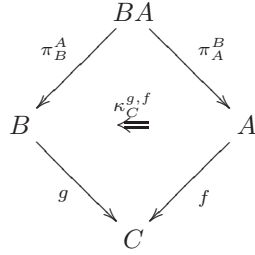
### Definition of Pullbacks

The main technical tool used in this paper is the pullback construction. This is used to define certain composites of spans and higher morphisms as part of the structure of the bicategory  $\text{Span}(\mathcal{B})_b$  and the tricategory  $\text{Span}(\mathcal{B})$ . We give an explicit definition of the limit and its universal property.

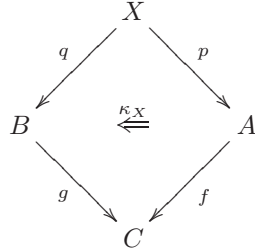
**Definition 1.** *Given a cospan*



*in a strict 2-category  $\mathcal{B}$ , the **pullback** is an object  $BA$ , equipped with projections  $\pi_A^B$ ,  $\pi_B^A$ , and a 2-cell  $\kappa_C^{g,f}$ :*



- *for any pair of maps  $p: X \rightarrow A$  and  $q: X \rightarrow B$  and 2-cell*



*there exists a unique 1-cell  $h: X \rightarrow BA$  such that*

$$p = \pi_A^B h, \quad q = \pi_B^A h, \quad \text{and} \quad \kappa_C^{g,f} \cdot h = \kappa_X, \quad \text{and}$$

- *for any pair of 1-cells  $j, k: X \rightarrow BA$  and 2-cells*

$$\varpi: \pi_A^B j \Rightarrow \pi_A^B k \quad \text{and} \quad \varrho: \pi_B^A j \Rightarrow \pi_B^A k,$$

*such that*

$$(g \cdot \varrho)(\kappa_C^{g,f} \cdot j) = (\kappa_C^{g,f} \cdot k)(f \cdot \varpi),$$

*there exists a unique 2-cell  $\gamma: j \Rightarrow k$  such that*

$$\pi_A^B \cdot \gamma = \varpi \quad \text{and} \quad \pi_B^A \cdot \gamma = \varrho.$$

### Definition of Finite Products

We define the product in  $\mathcal{B}$ .

**Definition 2.** *Given a pair of objects in a strict 2-category  $\mathcal{B}$ , the **product** is an object  $A \times B$  equipped with projections  $A \xrightarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$  such that:*

- *for each pair of maps  $p: X \rightarrow A$  and  $q: X \rightarrow B$ , there exists a unique 1-cell  $h: X \rightarrow A \times B$  such that*

$$p = \pi_A h \quad \text{and} \quad q = \pi_B h, \quad \text{and}$$

- for each pair of 1-cells  $j, k: X \rightarrow A \times B$  and 2-cells

$$\varpi: \pi_A j \Rightarrow \pi_A k \text{ and } \varrho: \pi_B j \Rightarrow \pi_B k,$$

there exists a unique 2-cell  $\gamma: j \Rightarrow k$  such that

$$\pi_A \cdot \gamma = \varpi \text{ and } \pi_B \cdot \gamma = \varrho.$$

We also include the nullary product, or terminal object as part of the structure of  $\mathcal{B}$ . In fact, a 2-category with iso-comma objects and a terminal object automatically has finite products, which are obtained by the obvious cospan with arrows into the terminal object.

**Definition 3.** We call an object  $1 \in \mathcal{B}$  the **terminal object** if for every object  $A \in \mathcal{B}$ , there is a unique 1-cell from  $A$  to  $1$ .

### 3 Monoidal Tricategories as One-Object Tetracategories

#### 3.1 Approaching a Definition

As the goal of this paper is to define a monoidal tricategory of spans and ‘monoidal tricategory’ is not a well-defined notion in the literature, we need to first specify what structure we have in mind. There is little doubt that a number of people either have or could write down a reasonable notion of monoidal structure on a tricategory if asked. In 1995, Trimble went a step further and wrote down a definition of tetracategory with axioms sprawling over dozens of pages [24]. Following the pattern of defining a monoidal category to be a one-object category one dimension above, we say

A monoidal tricategory is a one-object Trimble tetracategory.

In recalling Trimble’s definition we hopefully succeed in making tetracategories accessible to a wide audience. We explain Trimble’s notion of *product cells* for tritransformations and trimodifications, give a precise statement of the equivalence expected for structure cells at each level, and choose explicit 3-cells (geometric 2-cells in local tricategories) as the ‘interchange’ cells appearing in the tetracategory axioms. The choice of interchange cell is governed by coherence for tricategories, so all choices are suitably equivalent.

The definition of a tetracategory is largely straightforward. In fact, Trimble’s approach to defining tetracategories was to, as much as possible, formalize the process of drawing of the coherence axioms, at least up to coherent isomorphism. Just as monoids, or one-object categories, have associativity and unit axioms, higher categories have generalized associativity and unit coherence axioms. One starts by noting that the drawing of associativity axiom  $K_n$  is nearly canonical at each level  $n$ . These axioms first appeared as families of simplicial complexes called *associahedra* in work of Stasheff and called *orientals* in work of Street. These associator  $K_{n+2}$  axioms can, in turn, be used to define unit axioms  $U_{n+1,1}, \dots, U_{n+1,n+1}$  for weak  $n$ -categories.

It is useful to work up from the usual category axioms towards tetracategories developing intuition for the higher unit diagrams and building on successive steps. It can also be useful to think of categorical structure as consisting of both associativity and unit operations and axioms. The coherence axioms for categories include one associativity axiom  $K_3$ :

$$\otimes(\otimes \times 1) = \otimes(1 \times \otimes)$$

and two unit axioms  $U_{2,1}$ :

$$\otimes(I \times 1) = 1$$

and  $U_{2,2}$ :

$$\otimes(1 \times I) = 1.$$

Here the tensor product denotes the composition operation on the category and the three axioms often denoted  $\alpha$ ,  $\lambda$ , and  $\rho$ , respectively, for obvious reasons. We then recall that a category  $\mathcal{C}$  also has a unit operation  $I \in \text{Ob}(\mathcal{C})$ . Of course the unit object has an identity morphism, so we can write  $I$  as a functor to make it appear more like an operation:

$$I: 1 \rightarrow \mathcal{C}.$$

The unit operations and axioms are closely tied to those for associativity. We know that  $I$  is the unit for our composition operation  $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ . In fact,  $\otimes$  is the associativity operation in this case and has the associativity



axiom  $K_3$  as noted above. Although slightly awkward in many contexts it is useful here for combinatorial reasons to write  $K_2$  for the composition operation. Similarly, we can write  $U_1$  as the unit operation.

Formally, bicategories include both of the category operations  $K_2$  and  $U_2$ , which are now interpreted as functors between bicategories. The category axioms become bicategory operations:

$$K_3: K_2(K_2, 1) \Rightarrow K_2(1, K_2),$$

$$U_{2,1}: K_2(U_1, 1) \Rightarrow 1,$$

and

$$U_{2,2}: K_2(1, U_1) \Rightarrow 1.$$

The bicategory axioms are the MacLane pentagon, which we write algebraically as:

$$K_4: K_3(1, K_2) \circ K_3(K_2, 1) \Rightarrow K_2(1, K_3) \circ K_3(1, K_2) \circ K_2(K_3, 1).$$

Notice that each possible 4-ary operation with one occurrence of each  $K_2$  and  $K_3$  appears as a 1-cell (edge) of the pentagon. A similar pattern can be observed for the 0-cells (vertices) of the pentagon. We are left to derive the unit axioms from  $K_3$ .

We use  $K_3$  to construct a template informing the general shape of the unit axioms  $U_{3,1}$ ,  $U_{3,2}$ , and  $U_{3,3}$ . The second index  $i$  in  $U_{3,i}$  tells us that the unit object should appear in the  $i^{th}$  argument. The unit axiom  $U_{3,1}$  will have 1-cells in the domain resembling those in the domain of  $K_3$ , except that a copy of the unit object will be placed in the first argument of each operation. Similarly, the codomain will contain 1-cells from the codomain of  $K_3$  with units in the first argument. There is an extra associativity term appearing in the unit operation, which we now describe.

If we imagine an associativity operation for 0-categories,  $K_1: 1 \rightarrow 1$ , then we notice a pattern beginning with the bicategory unit operations. The domain and codomain of  $U_{2,1}$  contain the domain of and codomain of  $K_1$ , respectively. We notice the appearance of the associativity operation  $K_2$  in the domain. It turns out that this is a very general phenomenon. In each unit axiom  $U_{n,i}$  there is a cell  $K_n$  in the domain if  $n$  is odd or in the codomain if  $n$  is even. Finally, there is a copy of  $U_1$  appearing in the first and second arguments of  $K_2$  in  $U_{2,1}$  and  $U_{2,2}$ , respectively.

From the above considerations a pattern begins to become evident. We have bicategory axioms:

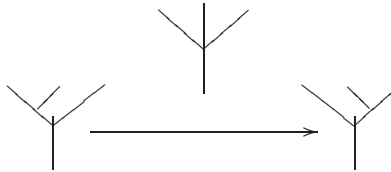
$$\text{Unit axiom } U_{3,1} \quad K_2(U_{2,1}, 1) = U_{2,1}(K_2) \circ K_3(U_1, 1, 1)$$

$$\text{Unit axiom } U_{3,2} \quad K_2(U_{2,2}, 1) = U_{2,2}(K_2) \circ K_3(1, U_1, 1)$$

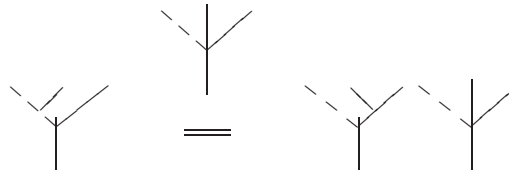
$$\text{Unit axiom } U_{3,3} \quad U_{2,2}(K_2) = K_2(1, U_{2,2}) \circ K_3(1, 1, U_1).$$

Trimble tames the complexity of his tetracategory axioms using operad-like trees to name the associativity and unit operations and axioms. While the construction outlined in the previous paragraph may not be entirely transparent in the unit axioms above, a brief explanation of the tree diagrams should provide clarity.

The  $K_3$  associativity operation is labelled by a “3-sprout” with domain and codomain the expected pair of rooted trees each having three leaves and one internal edge:



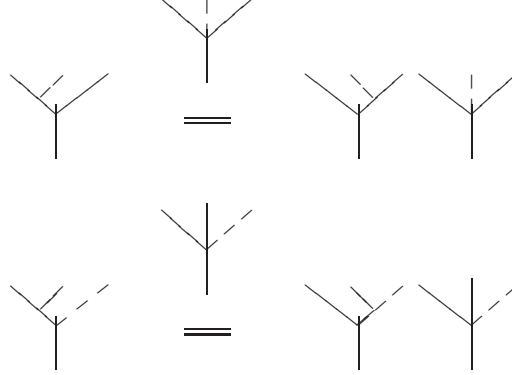
We can write down the unit axioms using trees this time. We replace solid edges with dashed edges according to the indices of the unit axiom and include the  $K_3$  factor as the 3-sprout with a dashed edge. We have, for example, the axiom  $U_{3,1}$ :



Now rewriting our unit operations as composites of  $K$  operations and the unit object, e.g.,  $U_{2,1} = K_2(U_1, 1)$ , we see the direct correspondence between the algebraic and tree descriptions of  $U_{3,1}$ :

$$U_{3,1}: K_2(K_2(U_1, 1), 1) \rightarrow K_2(U_1, K_2) \circ K_3(U_1, 1, 1).$$

Tree representations of the other unit axioms have the same underlying trees, so differ only by the dashed edge.



We have now written down three distinct unit axioms for bicategories, while the usual definition only requires the associativity axiom  $K_4$  and the single unit axiom  $U_{3,2}$ . This highlights a bit of history in the early development of higher category theory. When MacLane first defined monoidal categories, he included all three axioms (and some other axioms too). In 1964 Max Kelly proved that one unit triangle was sufficient. Thinking ahead one step we have the family of unit axioms for tricategories  $U_{4,1}$ ,  $U_{4,2}$ ,  $U_{4,3}$ , and  $U_{4,4}$ . However, while the definition of tricategory [10] contains all three structural unit operations  $U_{3,1}$ ,  $U_{3,2}$ ,  $U_{3,3}$ , only the two unit axioms  $U_{4,2}$  and  $U_{4,3}$  are required. Gurski remarks on this appearance of cells that are not lifted from equations one-categorical rung down the ladder of  $n$ -categories as an example of the appearance of interesting higher structure replacing equality [11].

Apart from the historical significance of the development of the unit axioms, it is also important in understanding Trimble's definition to highlight the relationship between the unit axioms. In particular, Trimble only defined three unit axioms  $U_{5,i}$ ,  $2 \leq i \leq 4$  conjecturing the continuation of the pattern for units.

**Conjecture 4** (Trimble [25]). *Given a non-negative integer  $n$ , the unit axioms  $U_{n,1}$  and  $U_{n,n}$  of weak  $n$ -categories follow from the associativity axiom  $K_{n+1}$ , the remaining unit axioms  $U_{n,i}$ ,  $2 \leq i \leq n-1$ , and the  $U_{n+1,j}$  unit axioms of a weak  $n+1$ -category.*

Given the above conjecture, Trimble omits the unit axioms  $U_{5,1}$  and  $U_{5,5}$  in the tetracategory definition. A proof would involve considering, at each categorical level, the structure of the unit operations and axioms one categorical dimension higher.

Having obtained all of the bicategory axioms we can move onto tricategories and tetracategories. These have associativity axioms  $K_5$  — the Stasheff polytope — and  $K_6$ , respectively. At this point, the reader is urged to write down the unit axioms  $U_{4,2}$  and  $U_{4,3}$  for tricategories. These diagrams appear in the definition below as perturbations along with  $U_{4,1}$  and  $U_{4,4}$ . Setting these operations to be identity perturbations, the tricategory unit axioms are recovered along with the equations  $U_{4,1}$  and  $U_{4,4}$ , which are redundant axioms for tetracategories.

At this point we should be able to understand unit axioms for tetracategories. These axioms are three-dimensional and can be rather intimidating at first glance, but can be understood very systematically with the aid of a few preliminary remarks. The cells on either side of the equations are each components of perturbations, trimodifications, or tritransformations. Trimble calls the component cells of trimodifications and tritransformations appearing in the axioms ‘product cells’. The tetracategory axioms are presented as equations between composites of geometric 3-cells (4-cells of the tetracategory) between surface diagrams.

The following discussion will use the modification  $m$  as an example. There is a similar story for associativity in which triangles are replaced by pentagons. A modification consists of a family of geometric 2-cells indexed by objects and a family of invertible modifications indexed by morphisms, which are naturality cells for the geometric 2-cells. The components of these naturality modifications are geometric 3-cells whose domains and codomains have factors corresponding to the domain and codomain of  $m$ , respectively, and modification 2-cells corresponding to the domain and codomain of the indexing morphism. In pictures, the ‘product’ is manifest. The unitor 2-cell is a

triangle:

$$\begin{array}{ccc}
(X \otimes 1) \otimes Y & \xrightarrow{\alpha_{X,1,Y}} & X \otimes (1 \otimes Y) \\
\rho_X \otimes 1_Y \searrow & \nearrow m_{XY} & \swarrow 1_X \otimes \lambda_Y \\
& X \otimes Y &
\end{array}$$

and, for morphisms  $(f, f') : (X, Y) \rightarrow (X', Y')$ , the modification 3-cell, which is the desired product 3-cell, fills a prism:

$$\begin{array}{ccccc}
& & (X' \otimes 1) \otimes Y' & \xrightarrow{\alpha_{X',1,Y'}} & X' \otimes (1 \otimes Y') \\
& \nearrow (f \otimes 1) \otimes f' & \nearrow \alpha_{f,1,f'} & \nearrow \rho_{X'} \otimes 1_{Y'} & \nearrow f \otimes (1 \otimes f') \\
(X \otimes 1) \otimes Y & \xrightarrow{\alpha_{X,1,Y}} & X \otimes (1 \otimes Y) & \xrightarrow{f \otimes 1} & X' \otimes (1 \otimes Y) \\
\rho_X \otimes 1_Y \searrow & \nearrow m_{XY} & \nearrow 1_X \otimes \lambda_Y & \nearrow 1_f \otimes \lambda_{f'} & \nearrow m_{X'Y'} \\
& X \otimes Y & \xrightarrow{f \otimes f'} & X' \otimes Y' &
\end{array}$$

We are left only to describe the three rectangular 2-cells. The three components of the domain and codomain of the triangle above correspond to the three rectangles. Two of the cells are in the domain of our 3-cell and the other is in the codomain. Finally, we said the prism should be the unitor triangle cross a  $K_3$ -interval, so  $(f, f')$  is either  $(1, \alpha)$  or  $(\alpha, 1)$ . There is a subtlety which is that when writing down these product cells for the middle modification in the axioms, for example, we defer to coherence for tricategories. The subtlety is in the labelling of certain cells, which we now explain.

When we denote a geometric 2-cell as the product of two structure cells, e.g.,  $\rho \times \alpha$ , it is at times useful to employ tricategorical coherence. Although, one may define these 2-cells by various composites, tricategorical coherence assures us that the resulting 2-cell diagrams are equivalent in the appropriate sense.

Consider, for example, the domain of the second cell in the composite of geometric 3-cells forming the domain of the  $U_{5,2}$  axiom. The domain of the 3-cell is the domain of a product cell of the middle mediator trimodification whiskered with various associativity tritransformation and trimodification cells. The 2-cell we now describe has the following form:

$$\rho \times \alpha^{-1} : (\rho_A \times 1_{B(CD)}) * (1_{A1} \times \alpha_{BCD}) \Rightarrow (1_A \times \alpha_{BCD}) * (\rho_A \times 1_{(BC)D}),$$

and it is evident that the composite is defined by a “higher-dimensional Eckmann-Hilton argument”.

The 2-cell  $\rho \times \alpha^{-1}$  is a square:

$$\begin{array}{ccc}
(xI)((yz)w) & \xrightarrow{1_x I * \alpha_{yzw}} & (xI)(y(zw)) \\
\rho_x * 1_{(yz)w} \downarrow & & \downarrow \rho_x * 1_{y(zw)} \\
x((yz)w) & \xrightarrow{1_x * \alpha_{yzw}} & x(y(zw))
\end{array}$$

Noticing that  $\rho$  and  $\alpha$  are acting, in some sense, independently of one another, we define this square as a composite of triangle 2-cells by introducing the diagonal 1-cell:

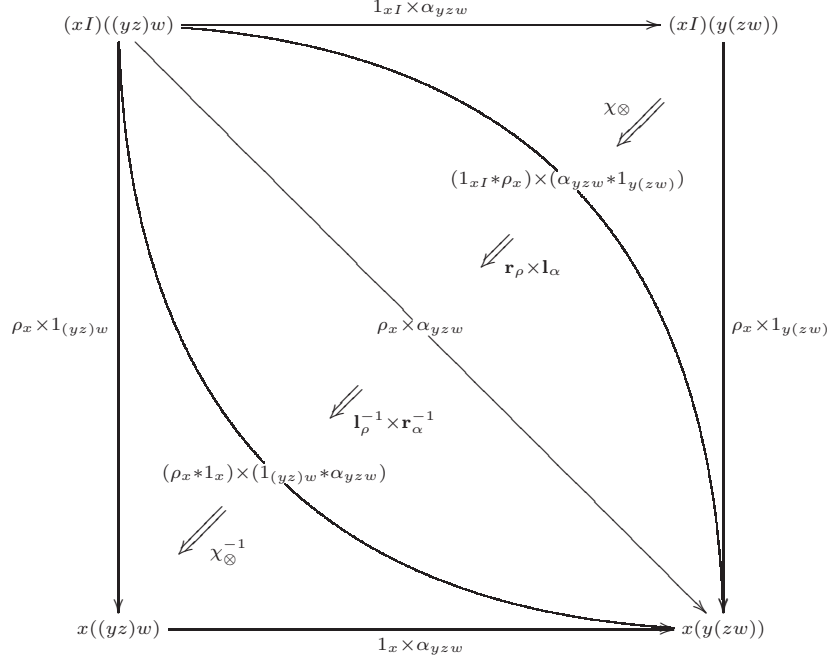
$$\rho_x * \alpha_{yzw} : (xI)((yz)w) \rightarrow x(y(zw)).$$

The 2-cells comprising the triangles are coherence cells for interchange and unit coherence cells. The interchange coherence cells are structure cells of the strong transformation component  $\chi_\otimes$  of the monoidal product trifunctor. The unit coherence cells are the adjoint pairs of 1-cells of adjoint equivalences in bicategories of 2-functors, strong transformations, and modifications, i.e., the 1-cells of the right and left unitor transformations of the local tricategories in which the diagrams of the  $U_{5,i}$  axioms live.

The composite is:

$$\rho \times \alpha^{-1} := \chi_{\otimes}^{-1}(l_{\rho}^{-1} \otimes r_{\alpha}^{-1})(r_{\rho} \otimes l_{\alpha})\chi_{\otimes}$$

which, for computational purposes, is best drawn as a square whose interior has been sectioned into a pair of bigons and a pair of triangles. We have:



Similar interchange-type cells are used to define  $\alpha^{-1} \widetilde{\times} \lambda$  and  $\alpha^{-1} \widetilde{\times} \alpha$  explicitly.

### 3.2 Trimble's Tetracategory Definition

The structure we work with in this paper is a monoidal tricategory.

**Definition 5.** A **monoidal tricategory** is a one-object tetracategory in the sense of Trimble.

We now give the definition of tetracategory following Trimble [24].

**Definition 6.** A **tetracategory**  $\mathcal{T}$  consists of:

- a collection of objects  $a, b, c, \dots$ ,
- for each pair of objects  $a, b$ , a tricategory  $\mathcal{T}(a, b)$  of 1-, 2-, and 3-morphisms,
- for each triple of objects  $a, b, c$ , a trifunctor:

$$\otimes: \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$$

called composition,

- for each object  $a$ , a trifunctor:

$$I: 1 \rightarrow \mathcal{T}$$

called a unit,

- for each 4-tuple of objects  $a, b, c, d$ , a biadjoint biequivalence (in the local tricategory of maps of tricategories):

$$\alpha: \otimes (\otimes \times 1) \Rightarrow \otimes (1 \times \otimes)$$

called the associativity,

- for each pair of objects  $a, b$ , a biadjoint biequivalence (in the local tricategory of maps of tricategories):

$$\lambda: \otimes (I \times 1) \Rightarrow 1$$

called the monoidal left unitor,

- for each pair of objects  $a, b$ , a biadjoint biequivalence (in the local tricategory of maps of tricategories):

$$\rho: \otimes (1 \times I) \Rightarrow 1$$

called the monoidal right unitor,

- for each 5-tuple of objects  $a, b, c, d, e$ , an adjoint equivalence (in the local bicategory of maps of tricategories):

$$\pi: (1 \times \alpha)\alpha(\alpha \times 1) \Rightarrow \alpha\alpha$$

called the pentagonator,

- for each triple of objects  $a, b, c$ , an adjoint equivalence (in the local bicategory of maps of tricategories):

$$l: (1 \times \lambda)\alpha \Rightarrow \lambda$$

called the left unit mediator,

- for each triple of objects  $a, b, c$ , an adjoint equivalence (in the local bicategory of maps of tricategories):

$$m: (\rho \times 1)\alpha \Rightarrow \lambda \times 1$$

called the middle unit mediator,

- for each triple of objects  $a, b, c$ , an adjoint equivalence (in the local bicategory of maps of tricategories):

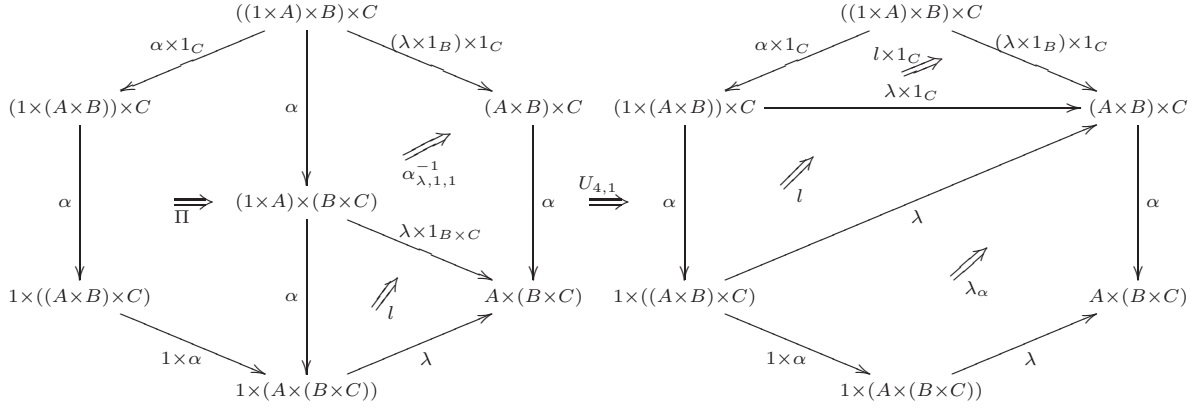
$$r: \rho\alpha \Rightarrow \rho \times 1$$

called the right unit mediator,

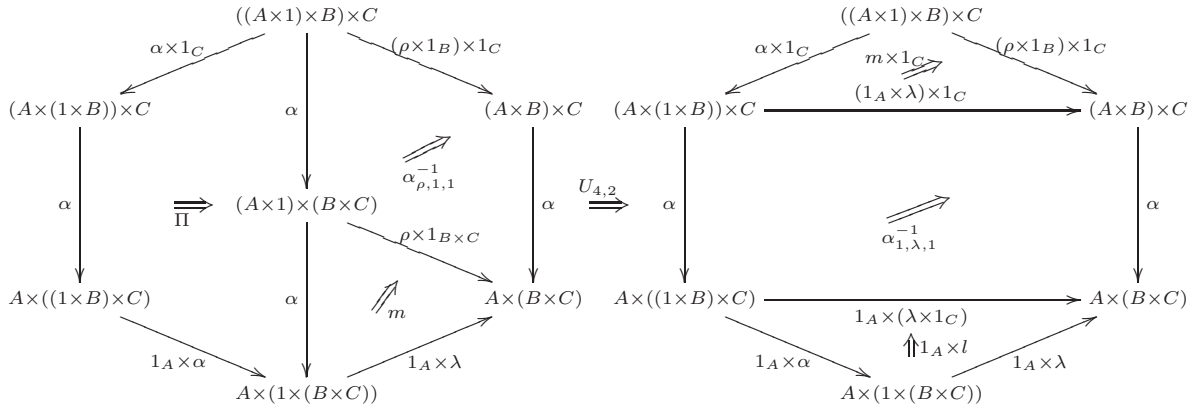
- for each 6-tuple of objects  $a, b, c, d, e, f$ , a perturbation called a (non-abelian) 4-cocycle (or  $K_5$ ), consisting of, for each 5-tuple of morphisms  $A, B, C, D, E$ , an invertible 4-cell:

The diagram illustrates the 4-cocycle  $K_5$  as a complex commutative diagram. It shows various compositions of morphisms  $A, B, C, D, E$  and the associator  $\alpha$ . The nodes represent different ways to parenthesize the product of these morphisms. The arrows represent the morphisms themselves, and the 4-cell  $K_5$  is shown as a large arrow connecting two different paths through the diagram. The diagram is a visual representation of the coherence of the associator  $\alpha$  in a tricategory.

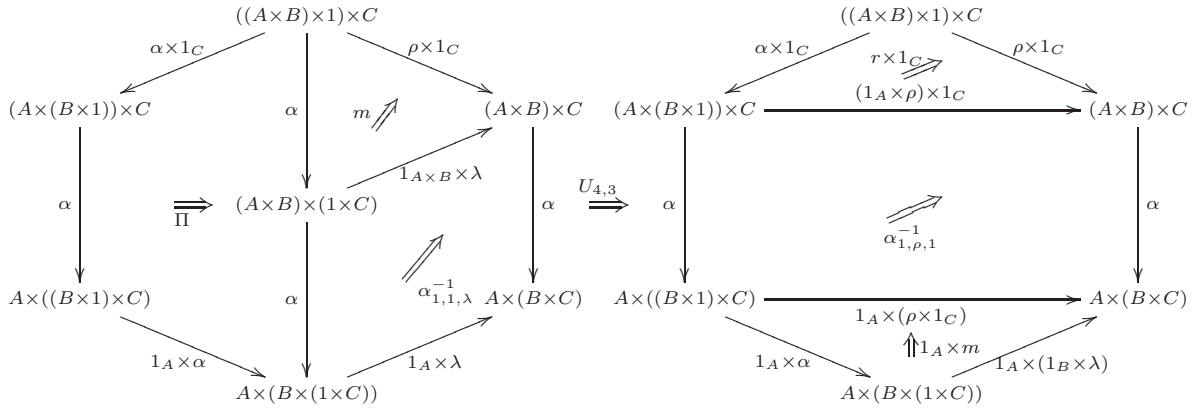
- for each 4-tuple of objects  $a, b, c, d$ , a perturbation called the  $U_{4,1}$  unit operation, consisting of, for each triple of morphisms  $A, B, C$ , an invertible 4-cell:



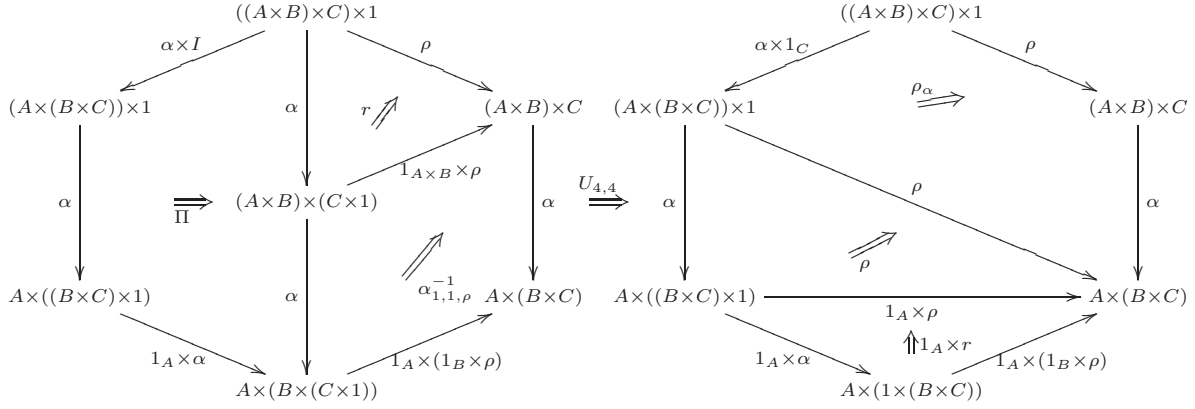
- for each 4-tuple of objects  $a, b, c, d$ , a perturbation called the  $U_{4,2}$  unit operation, consisting of, for each triple of morphisms  $A, B, C$ , an invertible 4-cell:



- for each 4-tuple of objects  $a, b, c, d$ , a perturbation called the  $U_{4,3}$  unit operation, consisting of, for each triple of morphisms  $A, B, C$ , an invertible 4-cell:



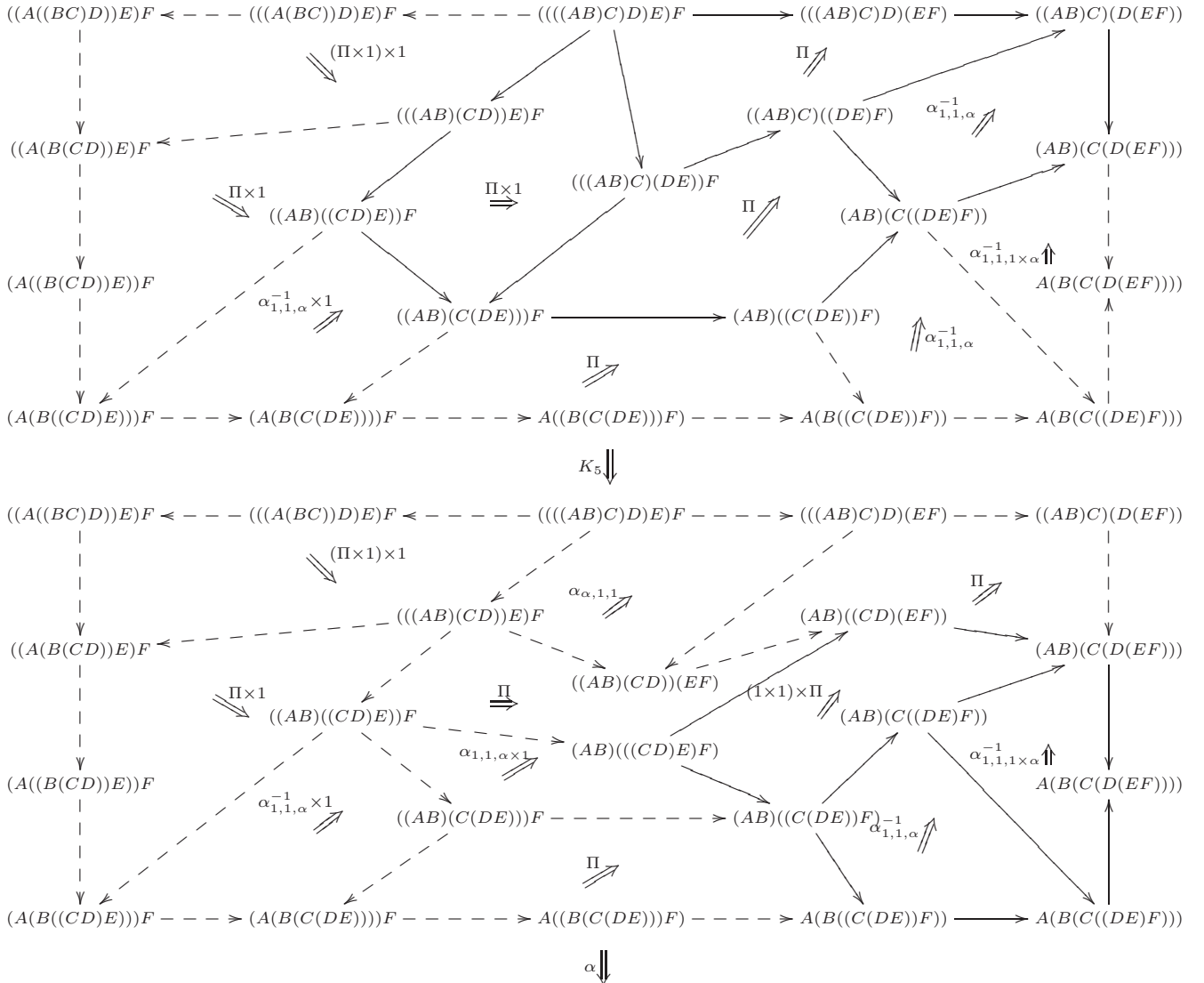
- for each 4-tuple of objects  $a, b, c, d$ , a perturbation called the  $U_{4,4}$  unit operation, consisting of, for each triple of morphisms  $A, B, C$ , an invertible 4-cell:

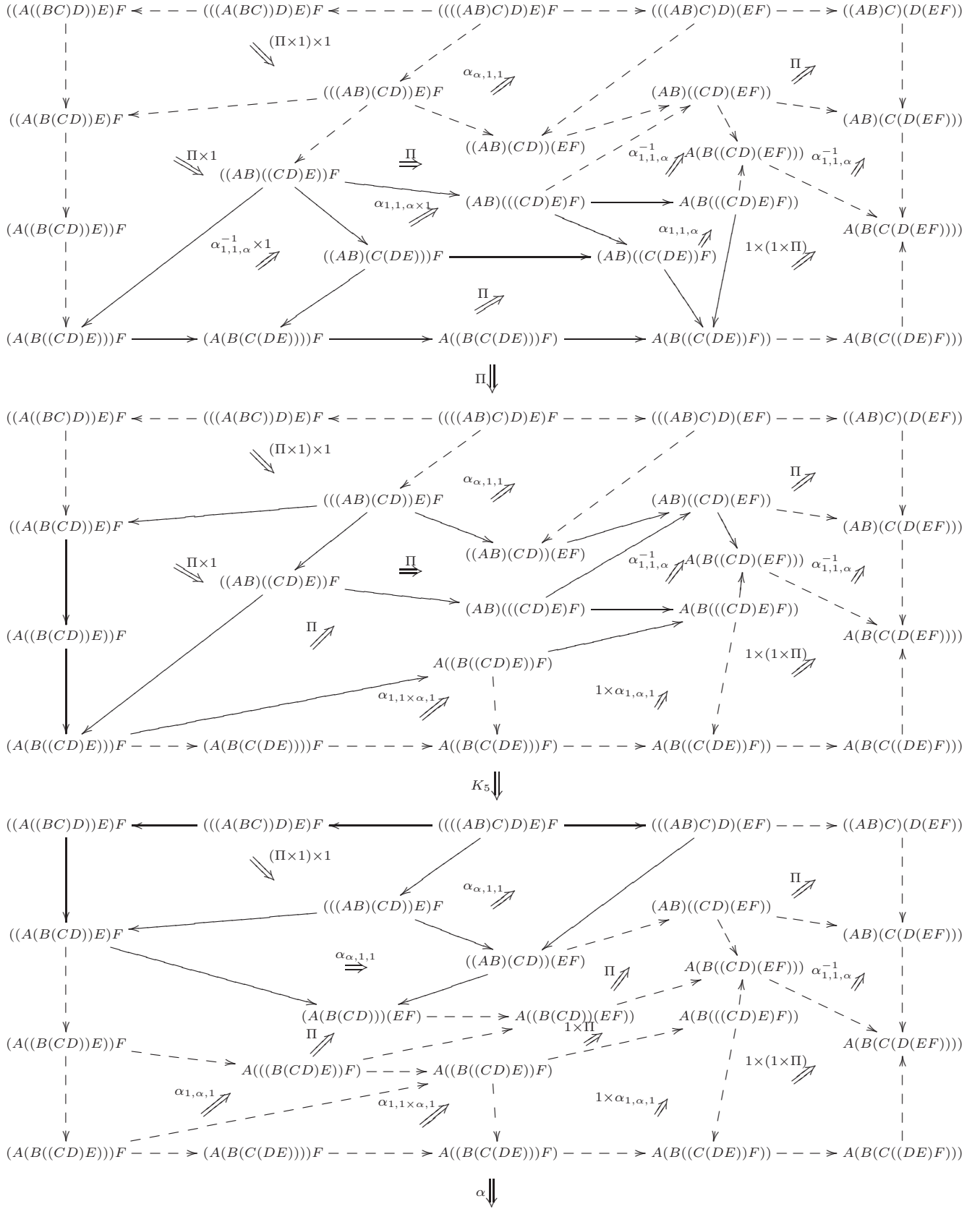


- all satisfying the  $K_6$  associativity condition and the  $U_{5,2}$ ,  $U_{5,3}$ , and  $U_{5,4}$  unit conditions:

### $K_6$ Axiom

- for each 7-tuple of objects  $a, b, c, d, e, f, g$ , an equation called the  $K_6$  associativity condition, consisting of, for each 6-tuple of morphisms  $A, B, C, D, E, F$ , an equation of 4-cells:







$$\begin{array}{c}
((A((BC)D))E)F \leftarrow \cdots \leftarrow (((A(BC))D)E)F \leftarrow \cdots \leftarrow (((AB)C)D)E)F \leftarrow \cdots \rightarrow (((AB)C)D)(EF) \rightarrow ((AB)C)(D(EF)) \\
\downarrow \quad \quad \quad \searrow^{\alpha_{\alpha,1,1}} \quad \quad \quad \swarrow^{\alpha_{\alpha \times 1,1,1}} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\Pi} \quad \quad \quad \downarrow \\
((A(B(CD))E)F \quad \quad \quad (A((BC)D))(EF) \leftarrow ((A(BC))D)(EF) \quad \quad \quad (AB)((CD)(EF)) \quad \quad \quad (AB)(C(D(EF))) \\
\downarrow \quad \quad \quad \searrow^{\alpha_{1 \times \alpha,1,1}} \quad \quad \quad \swarrow^{\Pi \times (1 \times 1)} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\alpha_{1,1,\alpha}^{-1}} \quad \quad \quad \downarrow \\
(A((B(CD))E))F \quad \quad \quad (A(B(CD)))(EF) \rightarrow A((B(CD))(EF)) \quad \quad \quad A(B((CD)(EF))) \quad \quad \quad A(B(C(D(EF)))) \\
\downarrow \quad \quad \quad \searrow^{\alpha_{1,\alpha,1}} \quad \quad \quad \swarrow^{\Pi} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\alpha_{1,1,\alpha}} \quad \quad \quad \downarrow \\
(A(B((CD)E)))F \quad \quad \quad A(((B(CD)E)F) \quad \quad \quad A((B((CD)E)F) \quad \quad \quad A(B(((CD)E)F)) \quad \quad \quad A(B(C(D(EF)))) \\
\downarrow \quad \quad \quad \searrow^{\alpha_{1,1 \times \alpha,1}} \quad \quad \quad \swarrow^{\alpha_{1,1 \times \alpha,1}} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\alpha_{1,1 \times \alpha,1}} \quad \quad \quad \downarrow \\
(A(B((CD)E)))F \quad \quad \quad (A(B(C(DE))))F \quad \quad \quad A((B(C(DE)))F) \quad \quad \quad A(B((C(DE))F)) \quad \quad \quad A(B(C((DE)F)))
\end{array}$$

$K_5 \Downarrow$

$$\begin{array}{c}
((A((BC)D))E)F \leftarrow \cdots \leftarrow (((A(BC))D)E)F \leftarrow \cdots \leftarrow (((AB)C)D)E)F \leftarrow \cdots \rightarrow (((AB)C)D)(EF) \rightarrow ((AB)C)(D(EF)) \\
\downarrow \quad \quad \quad \searrow^{\alpha_{\alpha,1,1}} \quad \quad \quad \swarrow^{\alpha_{\alpha \times 1,1,1}} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\alpha_{\alpha,1,1}} \quad \quad \quad \downarrow \\
((A(B(CD))E)F \quad \quad \quad (A((BC)D))(EF) \leftarrow ((A(BC))D)(EF) \quad \quad \quad (A(BC))(D(EF)) \quad \quad \quad (AB)(C(D(EF))) \\
\downarrow \quad \quad \quad \searrow^{\alpha_{1 \times \alpha,1,1}} \quad \quad \quad \swarrow^{\alpha_{1 \times \alpha,1,1}} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\alpha_{1,1,\alpha}^{-1}} \quad \quad \quad \downarrow \\
(A((B(CD))E))F \quad \quad \quad (A(B(CD)))(EF) \rightarrow A((B(CD))(EF)) \quad \quad \quad A(B((CD)(EF))) \quad \quad \quad A(B(C(D(EF)))) \\
\downarrow \quad \quad \quad \searrow^{\alpha_{1,\alpha,1}} \quad \quad \quad \swarrow^{\Pi} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\alpha_{1,1,\alpha}} \quad \quad \quad \downarrow \\
(A(B((CD)E)))F \quad \quad \quad A(((B(CD)E)F) \quad \quad \quad A((B((CD)E)F) \quad \quad \quad A(B(((CD)E)F)) \quad \quad \quad A(B(C(D(EF)))) \\
\downarrow \quad \quad \quad \searrow^{\alpha_{1,1 \times \alpha,1}} \quad \quad \quad \swarrow^{\alpha_{1,1 \times \alpha,1}} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\alpha_{1,1 \times \alpha,1}} \quad \quad \quad \downarrow \\
(A(B((CD)E)))F \quad \quad \quad (A(B(C(DE))))F \quad \quad \quad A((B(C(DE)))F) \quad \quad \quad A(B((C(DE))F)) \quad \quad \quad A(B(C((DE)F)))
\end{array}$$

$\equiv$

$$\begin{array}{c}
((A((BC)D))E)F \leftarrow \cdots \leftarrow (((A(BC))D)E)F \leftarrow \cdots \leftarrow (((AB)C)D)E)F \leftarrow \cdots \rightarrow (((AB)C)D)(EF) \rightarrow ((AB)C)(D(EF)) \\
\downarrow \quad \quad \quad \searrow^{\Pi \times 1 \times 1} \quad \quad \quad \swarrow^{\Pi \times 1} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\Pi} \quad \quad \quad \downarrow \\
((A(B(CD))E)F \quad \quad \quad ((AB)(CD))E)F \quad \quad \quad ((AB)C)(DE)F \quad \quad \quad ((AB)C)((DE)F) \quad \quad \quad (AB)(C(D(EF))) \\
\downarrow \quad \quad \quad \searrow^{\Pi \times 1} \quad \quad \quad \swarrow^{\Pi \times 1} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\alpha_{1,1,\alpha}^{-1}} \quad \quad \quad \downarrow \\
(A((B(CD))E))F \quad \quad \quad ((AB)((CD)E))F \quad \quad \quad ((AB)C)(DE)F \quad \quad \quad (AB)(C((DE)F)) \quad \quad \quad A(B(C(D(EF)))) \\
\downarrow \quad \quad \quad \searrow^{\alpha_{1,1,\alpha} \times 1} \quad \quad \quad \swarrow^{\Pi} \quad \quad \quad \downarrow \quad \quad \quad \searrow^{\alpha_{1,1,\alpha}} \quad \quad \quad \downarrow \\
(A(B((CD)E)))F \quad \quad \quad (A(B(C(DE))))F \quad \quad \quad A((B(C(DE)))F) \quad \quad \quad A(B((C(DE))F)) \quad \quad \quad A(B(C((DE)F)))
\end{array}$$

$K_5 \times 1 \Downarrow$

$$\begin{array}{c}
((A((BC)D))E)F \leftarrow \cdots \leftarrow (((A(BC))D)E)F \leftarrow \cdots \leftarrow (((AB)C)D)E)F \leftarrow \cdots \rightarrow (((AB)C)D)(EF) \leftarrow \cdots \rightarrow ((AB)C)(D(EF)) \\
\downarrow \quad \quad \quad \Pi \times 1 \quad \quad \quad \alpha_{\alpha,1,1} \times 1 \quad \quad \quad \Pi \nearrow \quad \quad \quad \alpha_{1,1,\alpha} \nearrow \\
((A(B(CD))E)F) \quad \quad \quad (A(((BC)D)E))F \quad \quad \quad ((A(BC))(DE))F \quad \quad \quad ((AB)C)((DE)F) \quad \quad \quad (AB)(C(D(EF))) \\
\downarrow \quad \quad \quad \alpha_{1,\alpha,1} \times 1 \quad \quad \quad \Pi \times 1 \quad \quad \quad \Pi \nearrow \quad \quad \quad \alpha_{1,1,\alpha} \nearrow \quad \quad \quad \downarrow \\
(A((B(CD))E))F \quad \quad \quad (A((BC)(DE)))F \quad \quad \quad ((AB)C)(DE)F \quad \quad \quad (AB)(C((DE)F)) \quad \quad \quad A(B(C(D(EF)))) \\
\downarrow \quad \quad \quad (1 \times \Pi) \times 1 \quad \quad \quad \Pi \nearrow \quad \quad \quad \alpha_{1,1,\alpha} \nearrow \quad \quad \quad \downarrow \\
(A(B((CD)E)))F \leftarrow \cdots \rightarrow (A(B(C(DE))))F \xrightarrow{\quad} A((B(C(DE))))F \xrightarrow{\quad} A(B((C(DE))F)) \xrightarrow{\quad} A(B(C((DE)F)))
\end{array}$$

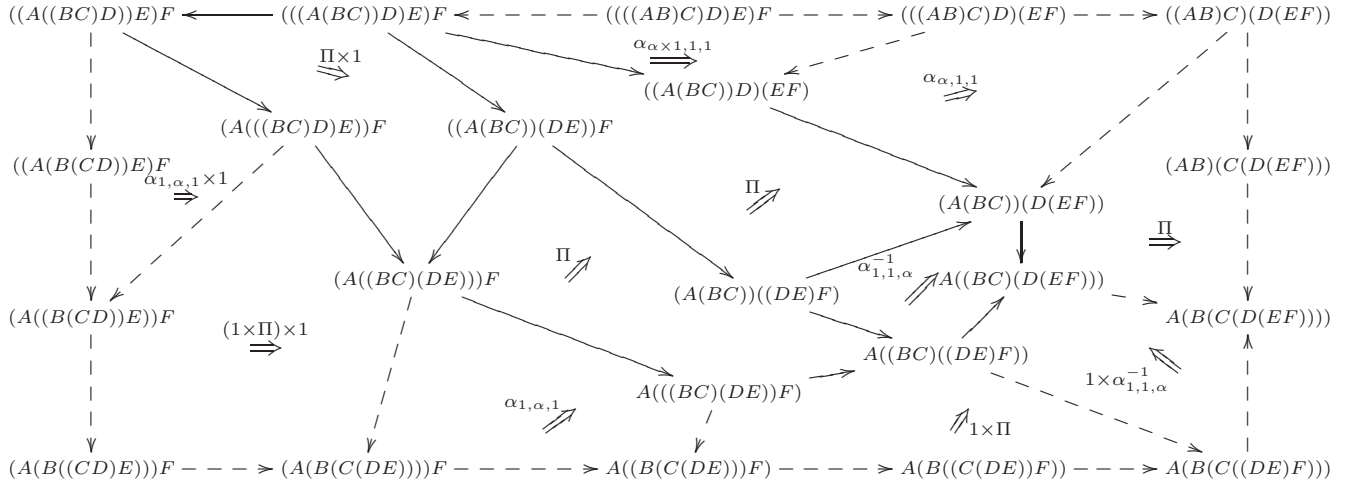
$K_5 \Downarrow$

$$\begin{array}{c}
((A((BC)D))E)F \leftarrow \cdots \leftarrow (((A(BC))D)E)F \leftarrow \cdots \leftarrow (((AB)C)D)E)F \leftarrow \cdots \rightarrow (((AB)C)D)(EF) \leftarrow \cdots \rightarrow ((AB)C)(D(EF)) \\
\downarrow \quad \quad \quad \Pi \times 1 \quad \quad \quad \alpha_{\alpha,1,1} \times 1 \quad \quad \quad \Pi \nearrow \quad \quad \quad \alpha_{1,1,\alpha}^{-1} \nearrow \\
((A(B(CD))E)F) \quad \quad \quad (A(((BC)D)E))F \quad \quad \quad ((A(BC))(DE))F \quad \quad \quad ((AB)C)((DE)F) \quad \quad \quad (AB)(C(D(EF))) \\
\downarrow \quad \quad \quad \alpha_{1,\alpha,1} \times 1 \quad \quad \quad \Pi \times 1 \quad \quad \quad \Pi \nearrow \quad \quad \quad \alpha_{1,1,\alpha}^{-1} \nearrow \quad \quad \quad \downarrow \\
(A((B(CD))E))F \quad \quad \quad (A((BC)(DE)))F \quad \quad \quad ((AB)C)(DE)F \quad \quad \quad (AB)(C((DE)F)) \quad \quad \quad A(B(C(D(EF)))) \\
\downarrow \quad \quad \quad (1 \times \Pi) \times 1 \quad \quad \quad \Pi \nearrow \quad \quad \quad \alpha_{1,1,\alpha}^{-1} \nearrow \quad \quad \quad \downarrow \\
(A(B((CD)E)))F \leftarrow \cdots \rightarrow (A(B(C(DE))))F \xrightarrow{\quad} A((B(C(DE))))F \xrightarrow{\quad} A(B((C(DE))F)) \xrightarrow{\quad} A(B(C((DE)F)))
\end{array}$$

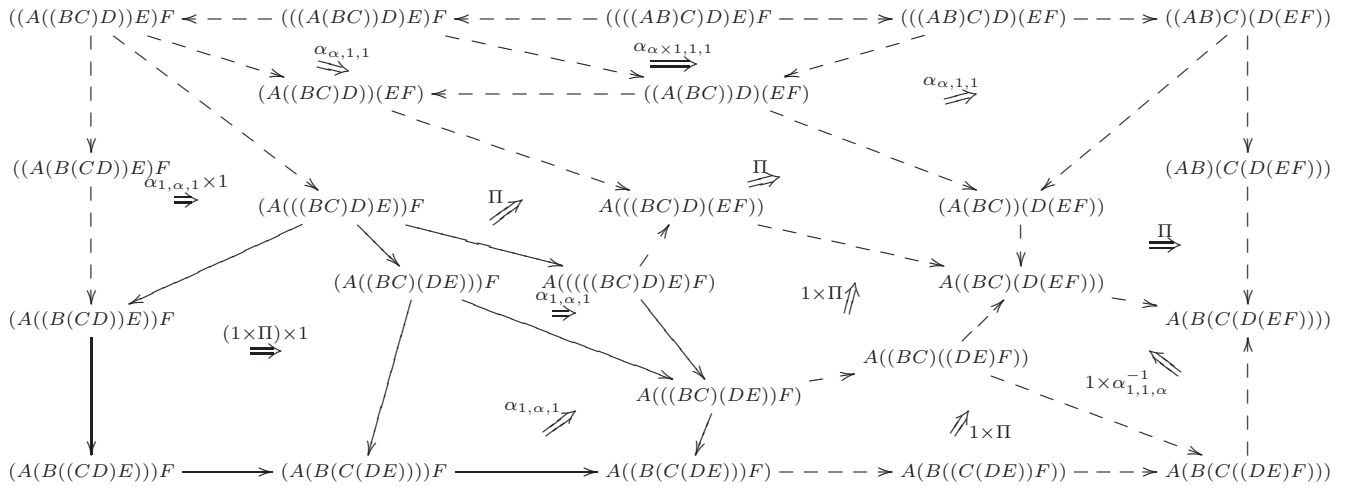
$\Pi \Downarrow$

$$\begin{array}{c}
((A((BC)D))E)F \leftarrow \cdots \leftarrow (((A(BC))D)E)F \xleftarrow{\quad} (((AB)C)D)E)F \xrightarrow{\quad} (((AB)C)D)(EF) \xrightarrow{\quad} ((AB)C)(D(EF)) \\
\downarrow \quad \quad \quad \Pi \times 1 \quad \quad \quad \alpha_{\alpha,1,1} \times 1 \quad \quad \quad \Pi \nearrow \quad \quad \quad \alpha^{-1} \tilde{\times} \alpha \nearrow \\
((A(B(CD))E)F) \quad \quad \quad (A(((BC)D)E))F \quad \quad \quad ((A(BC))(DE))F \quad \quad \quad ((AB)C)((DE)F) \quad \quad \quad (AB)(C(D(EF))) \\
\downarrow \quad \quad \quad \alpha_{1,\alpha,1} \times 1 \quad \quad \quad \Pi \times 1 \quad \quad \quad \Pi \nearrow \quad \quad \quad \alpha_{1,1,\alpha}^{-1} \nearrow \quad \quad \quad \downarrow \\
(A((B(CD))E))F \quad \quad \quad (A((BC)(DE)))F \quad \quad \quad ((AB)C)(DE)F \quad \quad \quad (AB)(C((DE)F)) \quad \quad \quad A(B(C(D(EF)))) \\
\downarrow \quad \quad \quad (1 \times \Pi) \times 1 \quad \quad \quad \Pi \nearrow \quad \quad \quad \alpha_{1,1,\alpha}^{-1} \nearrow \quad \quad \quad \downarrow \\
(A(B((CD)E)))F \leftarrow \cdots \rightarrow (A(B(C(DE))))F \xrightarrow{\quad} A((B(C(DE))))F \xrightarrow{\quad} A(B((C(DE))F)) \xrightarrow{\quad} A(B(C((DE)F)))
\end{array}$$

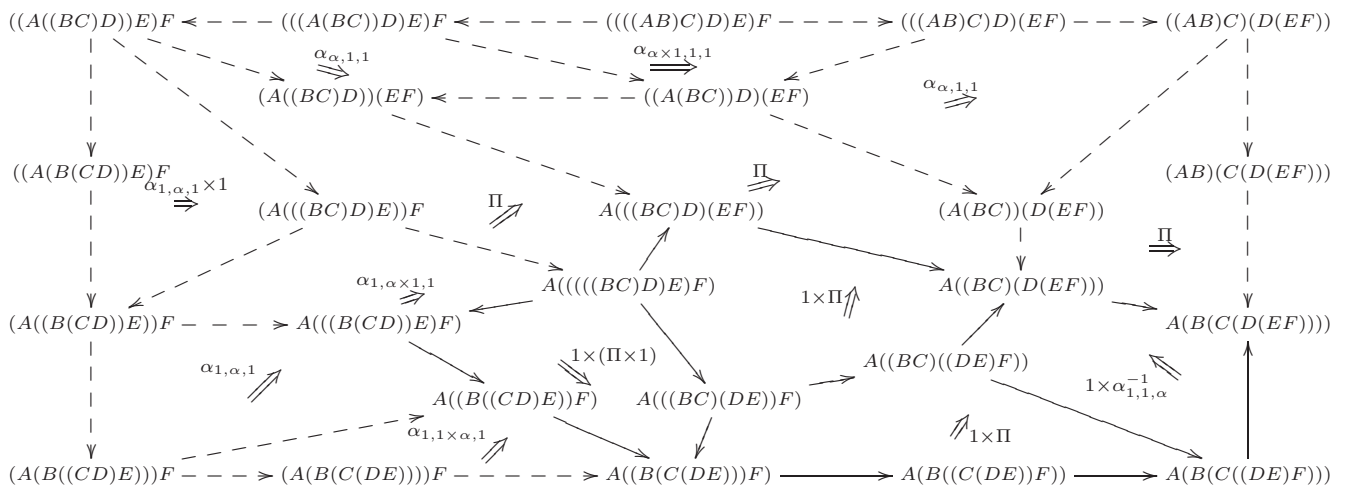
$\Pi \Downarrow$



$K_5 \Downarrow$



$\alpha \Downarrow$



$1 \times K_5 \Downarrow$





$$\begin{array}{c}
((A(1B))C)D \leftarrow (((A1)B)C)D \longrightarrow ((AB)C)D \quad ((A(1B))C)D \leftarrow \cdots \cdots (((A1)B)C)D \cdots \cdots \longrightarrow ((AB)C)D \\
\downarrow \quad \quad \quad \searrow \alpha_{\alpha,1,1}^{-1} \quad \quad \quad \nearrow \alpha_{\rho \times 1,1,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \searrow \alpha_{1 \times \lambda,1,1}^{-1} \quad \quad \quad \nearrow \alpha_{1,1,1}^{-1} \quad \quad \quad \downarrow \\
(A((1B)C))D \quad \quad \quad ((A1)B)(CD) \rightarrow (AB)(CD) \quad (A((1B)C))D \quad \quad \quad (AB)(CD) \\
\downarrow \quad \quad \quad \searrow \Pi \quad \quad \quad \nearrow m \times (1 \times 1) \quad \quad \quad \downarrow \quad \quad \quad \searrow \Pi \quad \quad \quad \nearrow \alpha \\
A(((1B)C)D) \quad (A(1B))(CD) \quad \quad \quad A(B(CD)) \quad A(((1B)C)D) \quad (A(1B))(CD) \quad \quad \quad A(B(CD)) \\
\downarrow \quad \quad \quad \searrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \searrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad \downarrow \\
A((1B)(CD)) \quad \quad \quad A(1(B(CD))) \quad \quad \quad A(1(B(CD))) \quad A((1B)(CD)) \quad \quad \quad A(1(B(CD))) \quad \quad \quad A(1(B(CD))) \\
\downarrow \quad \quad \quad \searrow 1 \times \Pi \quad \quad \quad \nearrow 1 \times l \quad \quad \quad \downarrow \quad \quad \quad \searrow 1 \times \Pi \quad \quad \quad \nearrow 1 \times l \quad \quad \quad \downarrow \\
A((1(BC))D) \cdots \cdots \rightarrow A(1((BC)D)) \cdots \cdots \rightarrow A(1(B(CD))) \quad A((1(BC))D) \cdots \cdots \rightarrow A(1((BC)D)) \cdots \cdots \rightarrow A(1(B(CD)))
\end{array}$$

$\Pi$

$$\begin{array}{c}
((A(1B))C)D \leftarrow \cdots \cdots (((A1)B)C)D \cdots \cdots \longrightarrow ((AB)C)D \quad ((A(1B))C)D \leftarrow \cdots \cdots (((A1)B)C)D \cdots \cdots \longrightarrow ((AB)C)D \\
\downarrow \quad \quad \quad \searrow (m \times 1) \times 1 \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \times 1 \quad \quad \quad \downarrow \quad \quad \quad \searrow (m \times 1) \times 1 \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \times 1 \quad \quad \quad \downarrow \\
(A((1B)C))D \cdots \cdots \rightarrow (A(BC))D \quad \quad \quad (AB)(CD) \quad (A((1B)C))D \cdots \cdots \rightarrow (A(BC))D \quad \quad \quad (AB)(CD) \\
\downarrow \quad \quad \quad \searrow \alpha_{1,\lambda \times 1,1}^{-1} \quad \quad \quad \nearrow 1 \times U_{4,1} \quad \quad \quad \downarrow \quad \quad \quad \searrow \alpha_{1,\lambda \times 1,1}^{-1} \quad \quad \quad \nearrow 1 \times U_{4,1} \quad \quad \quad \downarrow \\
A(((1B)C)D) \quad \quad \quad A((BC)D) \quad \quad \quad A(B(CD)) \quad A(((1B)C)D) \quad \quad \quad A((BC)D) \quad \quad \quad A(B(CD)) \\
\downarrow \quad \quad \quad \searrow 1 \times \alpha_{\lambda,1,1}^{-1} \quad \quad \quad \nearrow 1 \times (\lambda \times 1) \quad \quad \quad \downarrow \quad \quad \quad \searrow 1 \times \alpha_{\lambda,1,1}^{-1} \quad \quad \quad \nearrow 1 \times (\lambda \times 1) \quad \quad \quad \downarrow \\
A((1B)(CD)) \quad \quad \quad A(1(B(CD))) \quad \quad \quad A(1(B(CD))) \quad A((1B)(CD)) \quad \quad \quad A(1(B(CD))) \quad \quad \quad A(1(B(CD))) \\
\downarrow \quad \quad \quad \searrow 1 \times \Pi \quad \quad \quad \nearrow 1 \times \lambda \quad \quad \quad \downarrow \quad \quad \quad \searrow 1 \times \Pi \quad \quad \quad \nearrow 1 \times \lambda \quad \quad \quad \downarrow \\
A((1(BC))D) \longrightarrow A(1((BC)D)) \longrightarrow A(1(B(CD))) \quad A((1(BC))D) \longrightarrow A(1((BC)D)) \longrightarrow A(1(B(CD)))
\end{array}$$

### $U_{5,3}$ Axiom

- for each 5-tuple of objects  $a, b, c, d, e$ , an equation called the  $U_{5,3}$  unit condition, consisting of, for each 4-tuple of morphisms  $A, B, C, D$ , an equation of 4-cells:

$$\begin{array}{c}
((A(B1))C)D \leftarrow \cdots \cdots (((AB)1)C)D \longrightarrow ((AB)C)D \quad ((A(B1))C)D \leftarrow \cdots \cdots (((AB)1)C)D \cdots \cdots \longrightarrow ((AB)C)D \\
\downarrow \quad \quad \quad \searrow \Pi \times 1 \quad \quad \quad \nearrow \alpha_{\rho,1,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \searrow \Pi \times 1 \quad \quad \quad \nearrow m \times 1 \quad \quad \quad \downarrow \\
(A((B1)C))D \quad \quad \quad ((AB)(1C))D \quad \quad \quad ((AB)1)(CD) \rightarrow (AB)(CD) \quad (A((B1)C))D \quad \quad \quad ((AB)(1C))D \quad \quad \quad (AB)(CD) \\
\downarrow \quad \quad \quad \searrow \Pi \quad \quad \quad \nearrow m \quad \quad \quad \downarrow \quad \quad \quad \searrow \Pi \quad \quad \quad \nearrow 1 \times U_{4,2} \quad \quad \quad \downarrow \\
A(((B1)C)D) \quad \quad \quad (AB)((1C)D) \quad \quad \quad (AB)(1(CD)) \quad \quad \quad A(B(CD)) \quad A(((B1)C)D) \quad \quad \quad (AB)((1C)D) \quad \quad \quad (AB)(1(CD)) \quad \quad \quad A(B(CD)) \\
\downarrow \quad \quad \quad \searrow \alpha_{1,\alpha,1}^{-1} \quad \quad \quad \nearrow \alpha_{1,1,\alpha}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \searrow \alpha_{1,\alpha,1}^{-1} \quad \quad \quad \nearrow \alpha_{1,1,\alpha}^{-1} \quad \quad \quad \downarrow \\
A((B(1C))D) \cdots \cdots \rightarrow A(B((1C)D)) \cdots \cdots \rightarrow A(B(1(CD))) \quad A((B(1C))D) \cdots \cdots \rightarrow A(B((1C)D)) \cdots \cdots \rightarrow A(B(1(CD)))
\end{array}$$

$\alpha$

$$\begin{array}{c}
((A(B1))C)D \leftarrow \cdots \leftarrow (((AB)1)C)D \cdots \rightarrow ((AB)C)D \quad ((A(B1))C)D \leftarrow \cdots \leftarrow (((AB)1)C)D \cdots \rightarrow ((AB)C)D \\
\downarrow \quad \quad \quad \nearrow \Pi \times 1 \quad \quad \quad \nearrow m \times 1 \quad \quad \quad \downarrow \quad \quad \quad \nearrow \Pi \times 1 \quad \quad \quad \nearrow m \times 1 \quad \quad \quad \downarrow \\
(A((B1)C))D \quad ((AB)(1C))D \quad \quad \quad (AB)(CD) \quad (A((B1)C))D \quad ((AB)(1C))D \quad \quad \quad (A(BC))D \quad \quad \quad (AB)(CD) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \times 1 \quad \quad \quad \downarrow \quad \quad \quad \nearrow \Pi \\
A(((B1)C)D) \Rightarrow (A(B(1C)))D \quad \quad \quad (AB)((1C)D) \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad A((BC)D) \quad \quad \quad (AB)(CD) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{1,\alpha,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \times 1 \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{1,\alpha,1}^{-1} \\
A((B(1C))D) \Rightarrow A(B(1C))D \quad \quad \quad \nearrow \Pi \quad \quad \quad A(B(CD)) \quad \quad \quad A(((B1)C)D) \Rightarrow (A(B(1C)))D \quad \quad \quad A((BC)D) \quad \quad \quad A(B(CD)) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \nearrow 1 \times (1 \times l) \quad \quad \quad \downarrow \quad \quad \quad \nearrow 1 \times \alpha_{1,\lambda,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \nearrow 1 \times (1 \times l) \\
A((B(1C))D) \rightarrow A(B((1C)D)) \cdots \rightarrow A(B(1(CD)))A((B(1C))D) \cdots \rightarrow A(B((1C)D)) \cdots \rightarrow A(B(1(CD)))
\end{array}$$

$U_{4,3} \times 1$

$$\begin{array}{c}
((A(B1))C)D \leftarrow \cdots \leftarrow (((AB)1)C)D \cdots \rightarrow ((AB)C)D \quad ((A(B1))C)D \leftarrow \cdots \leftarrow (((AB)1)C)D \cdots \rightarrow ((AB)C)D \\
\downarrow \quad \quad \quad \nearrow (r \times 1) \times 1 \quad \quad \quad \nearrow \alpha_{1,\rho,1}^{-1} \times 1 \quad \quad \quad \downarrow \quad \quad \quad \nearrow (r \times 1) \times 1 \quad \quad \quad \nearrow \alpha_{1,\rho,1}^{-1} \times 1 \quad \quad \quad \downarrow \\
(A((B1)C))D \rightarrow (A(BC))D \quad \quad \quad (AB)(CD) \quad (A((B1)C))D \rightarrow (A(BC))D \quad \quad \quad (AB)(CD) \\
\downarrow \quad \quad \quad \nearrow (1 \times m) \times 1 \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{1,\rho,1}^{-1} \times 1 \quad \quad \quad \downarrow \quad \quad \quad \nearrow \Pi \\
A(((B1)C)D) \Rightarrow (A(B(1C)))D \quad \quad \quad \nearrow \alpha_{1,\alpha,1}^{-1} \quad \quad \quad A((BC)D) \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad A((BC)D) \quad \quad \quad (AB)(CD) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \nearrow 1 \times \alpha_{1,\lambda,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \nearrow 1 \times (m \times 1) \quad \quad \quad \downarrow \quad \quad \quad \nearrow 1 \times \alpha_{1,\lambda,1}^{-1} \\
A((B(1C))D) \Rightarrow A(B((1C)D)) \cdots \rightarrow A(B(1(CD)))A((B(1C))D) \cdots \rightarrow A(B((1C)D)) \cdots \rightarrow A(B(1(CD)))
\end{array}$$

$\alpha$

$=$

$$\begin{array}{c}
((A(B1))C)D \leftarrow \cdots \leftarrow (((AB)1)C)D \cdots \rightarrow ((AB)C)D \quad ((A(B1))C)D \leftarrow \cdots \leftarrow (((AB)1)C)D \cdots \rightarrow ((AB)C)D \\
\downarrow \quad \quad \quad \nearrow \Pi \times 1 \quad \quad \quad \nearrow \alpha_{\rho,1,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{\alpha,1,1}^{-1} \quad \quad \quad \nearrow \alpha_{\rho,1,1}^{-1} \quad \quad \quad \downarrow \\
(A((B1)C))D \quad ((AB)(1C))D \quad \quad \quad ((AB)1)(CD) \rightarrow (AB)(CD) \quad (A((B1)C))D \quad ((AB)1)(CD) \rightarrow (AB)(CD) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \nearrow \Pi \quad \quad \quad \downarrow \quad \quad \quad \nearrow m \quad \quad \quad \downarrow \quad \quad \quad \nearrow \Pi \quad \quad \quad \downarrow \quad \quad \quad \nearrow m \\
A(((B1)C)D) \Rightarrow (A(B(1C)))D \quad \quad \quad (AB)((1C)D) \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad A((BC)D) \quad \quad \quad (AB)(1(CD)) \Rightarrow A(B(CD)) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{1,\alpha,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \nearrow \alpha_{1,\lambda,1}^{-1} \quad \quad \quad \downarrow \quad \quad \quad \nearrow \Pi \\
A((B(1C))D) \Rightarrow A(B((1C)D)) \quad \quad \quad \nearrow \Pi \quad \quad \quad A(B(1(CD)))A((B(1C))D) \cdots \rightarrow A(B((1C)D)) \cdots \rightarrow A(B(1(CD)))
\end{array}$$

$K_5$

$U_{4,3}$







The top diagram is a complex commutative diagram representing the coherence of the tricategory of spans. It features several nodes representing compositions of spans and 2-cells. The nodes are arranged in a grid-like fashion, with some nodes connected by solid arrows and others by dashed arrows. The arrows are labeled with 1-cells and 2-cells. For example, the top row of nodes is  $((A(BC))1)D$ ,  $((AB)C)1D$ ,  $((AB)C)D$ ,  $((A(BC))1)D$ ,  $((AB)C)1D$ , and  $((AB)C)D$ . The bottom row of nodes is  $A((B(C1))D)$ ,  $A(B((C1)D))$ ,  $A(B(C(1D)))$ ,  $A((B(C1))D)$ ,  $A(B((C1)D))$ , and  $A(B(C(1D)))$ . The diagram is labeled  $U_{4,3}$  at the bottom.

The bottom diagram is another complex commutative diagram, similar to the top one, but with different 2-cells. It also features nodes representing compositions of spans and 2-cells. The nodes are arranged in a grid-like fashion, with some nodes connected by solid arrows and others by dashed arrows. The arrows are labeled with 1-cells and 2-cells. For example, the top row of nodes is  $((A(BC))1)D$ ,  $((AB)C)1D$ ,  $((AB)C)D$ ,  $((A(BC))1)D$ ,  $((AB)C)1D$ , and  $((AB)C)D$ . The bottom row of nodes is  $A((B(C1))D)$ ,  $A(B((C1)D))$ ,  $A(B(C(1D)))$ ,  $A((B(C1))D)$ ,  $A(B((C1)D))$ , and  $A(B(C(1D)))$ . The diagram is labeled  $1 \times U_{4,3}$  in the center.

## 4 A Tricategory of Spans

We now restate the main theorem of this section followed by a series of definitions and propositions, which together prove the result.

**Theorem 7.** *Let  $\mathcal{B}$  be a (strict) 2-category with pullbacks, consisting of:*

- 0-cells  $A, B, C, \dots$ ,
- 1-cells  $p, q, r, \dots$ , and
- 2-cells  $\varpi, \varrho, \varsigma, \dots$

*There is a semi-strict cubical tricategory  $\text{Span}(\mathcal{B})$  consisting of:*

- the objects of  $\mathcal{B}$  as objects,
- for each pair of objects  $A, B$ , the (strict) 2-category  $\text{Span}(A, B)$ , defined in Proposition 23, consisting of:
  - spans in  $\mathcal{B}$  (see Definition 9),
  - maps of spans in  $\mathcal{B}$  (see Definition 10), and
  - maps of maps of spans in  $\mathcal{B}$  (see Definition 11),

*which are, respectively, the 1-morphisms, 2-morphisms, and 3-morphisms of  $\text{Span}(\mathcal{B})$ ,*

- for each triple of objects  $A, B, C$ , a strict 2-functor:

$$*_{ABC}: \text{Span}(A, B) \otimes \text{Span}(B, C) \rightarrow \text{Span}(A, C),$$

defined in Proposition 25,

- for each object  $A$ , a strict 2-functor:

$$I_A: 1 \rightarrow \text{Span}(A, A),$$

defined in Proposition 27,

- for each 4-tuple  $A, B, C, D$  of objects, a strict adjoint equivalence:

$$\mathbf{a}_{ABCD}: *_{(AB)CD} (*_{ABC} \times 1) \Rightarrow *_{A(BC)D} (1 \times *_{BCD})$$

in the 2-category of strict 2-functors, strict transformations, and modifications, defined in Proposition 28,

- for each pair of objects  $A, B$ , an adjoint equivalence:

$$\mathbf{l}_{AB}: *_{ABB} (I_B \times 1) \Rightarrow 1,$$

in the 2-category of strict 2-functors, strong transformations, and modifications, defined in Proposition 29,

- for each pair of objects  $A, B$ , an adjoint equivalence:

$$\mathbf{r}_{AB}: *_{AAB} (1 \times I_A) \Rightarrow 1,$$

in the 2-category of strict 2-functors, strong transformations, and modifications, defined in Proposition 30,

- for each 5-tuple of objects  $A, B, C, D, E$ , an identity modification:

$$\Pi_{ABCDE}: (* \cdot (1 \times \mathbf{a}))(\mathbf{a} \cdot (1 \times * \times 1))(* \cdot (\mathbf{a} \times 1)) \Rightarrow (\mathbf{a} \cdot (1 \times 1 \times *))(* \cdot 1)(\mathbf{a} \cdot (1 \times 1 \times 1)),$$

defined in Proposition 31,

- for each triple of objects  $A, B, C$ , an identity modification:

$$\Lambda_{ABC}: 1(* \cdot (1 \times 1)) \Rightarrow (\mathbf{l} \cdot *) (\mathbf{a} \cdot (1 \times 1 \times I))(* \cdot 1),$$

defined Proposition 32,

- for each triple of objects  $A, B, C$ , an identity modification:

$$M_{ABC}: (* \cdot (\mathbf{l} \times 1))(\mathbf{a}^{-1} \cdot (1 \times I \times 1))(* \cdot (1 \times \mathbf{r}^{-1})) \Rightarrow * \cdot 1,$$

defined in Proposition 33, and

- for each triple of objects  $A, B, C$ , an identity modification:

$$P_{ABC}: (* \cdot (1 \times \mathbf{r}))1 \Rightarrow (\mathbf{r} \cdot *) (* \cdot 1)(\mathbf{a} \cdot (I \times 1 \times 1)),$$

defined Proposition 34.

*Proof.* The structural components are given in the referenced definitions and propositions. The tricategory is cubical since it is locally strict with strict, thus cubical, composition and unit 2-functors, and all product cells and modification components are identity morphisms. The tricategory axioms then follow and we have the desired result.  $\square$

**Remark 8.** If the adjoint equivalences  $\mathbf{a}, \mathbf{l}, \mathbf{r}$  were identities, then  $\text{Span}(\mathcal{B})$  would have the structure of a Gray-category. We are using the terminology semi-strict to reflect the fact that the tricategory has only trivial modifications, but possibly non-trivial adjoint equivalences.

## 4.1 Span Morphisms

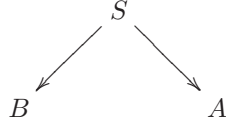
We define the morphisms of the span tricategory and set notation for the pullbacks and products used in defining the structure cells of the tricategory and the monoidal structure on the tricategory.

Let  $\mathcal{B}$  be a strict 2-category. The morphisms of  $\mathcal{B}$  are called 1-cells and 2-cells. The structure cells of the span tricategory  $\text{Span}(\mathcal{B})$  are called (1-)morphisms, 2-morphisms, and 3-morphisms. We define the morphisms of  $\text{Span}(\mathcal{B})$  here.

### Definitions of morphisms

The 1-morphisms in  $\text{Span}(\mathcal{B})$  are ‘spans’.

**Definition 9.** A **span** in a 2-category  $\mathcal{B}$  is a pair of 1-cells in  $\mathcal{B}$  with a common source object:

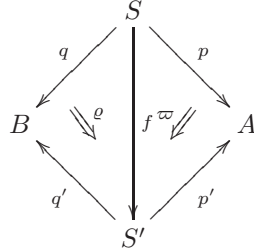


The 2-morphisms in  $\text{Span}(\mathcal{B})$  are ‘maps of spans’.

**Definition 10.** Given a pair of parallel spans in a 2-category  $\mathcal{B}$ :

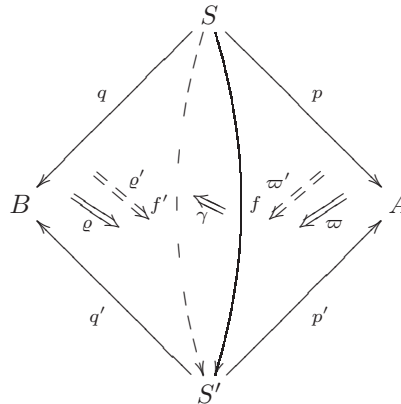


a **map of spans** is a triple  $(\varpi, f, \varrho)$  consisting of a 1-cell  $f: S \rightarrow S'$  together with a pair of invertible 2-cells  $\alpha: p \Rightarrow p'f$  and  $\beta: q \Rightarrow q'f$ :



The 3-morphisms in  $\text{Span}(\mathcal{B})$  are ‘maps of maps of spans’.

**Definition 11.** Given a parallel pair of maps of spans  $(\varpi, f, \varrho)$  and  $(\varpi', f', \varrho')$  in a 2-category  $\mathcal{B}$ :



a **map of maps of spans**  $\gamma: (\varpi, f, \varrho) \Rightarrow (\varpi', f', \varrho')$  consists of a 2-cell  $\gamma: f \Rightarrow f'$  in  $\mathcal{B}$  such that the following equations hold:

$$(p' \cdot \gamma)\varpi = \varpi' \quad (1)$$

and

$$(q' \cdot \gamma)\varrho = \varrho' \quad (2)$$

The notation  $\varpi \cdot f$  denotes the *whiskering* of a 2-cell  $\varpi$  along a 1-cell  $f$ . This is defined by horizontal composition of the 2-cell  $\varpi$  with the identity 2-cell  $1_f$  in  $\mathcal{B}$ .

By definition a map of maps of spans  $\gamma: (\varpi, f, \varrho) \Rightarrow (\varpi', f', \varrho')$  satisfies the equations:

$$(p' \cdot \gamma)\varpi = \varpi' \text{ and } (q' \cdot \gamma)\varpi = \varrho'.$$

If there exists a 2-cell  $\gamma^{-1}$ , we can compose on the left by  $p' \cdot \gamma$  and  $q' \cdot \gamma$ , respectively, to obtain

$$(p' \cdot \gamma^{-1})\varpi' = \varpi \text{ and } (q' \cdot \gamma^{-1})\varrho' = \varrho.$$

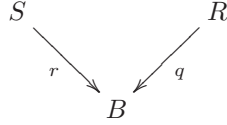
It follows that  $\gamma^{-1}: (\varpi', f', \varrho') \Rightarrow (\varpi, f, \varrho)$  is a map of maps of spans as well.

**Remark 12.** The maps of spans in Definition 10 are the 2-morphisms in our construction. In groupoidification, and, more generally, geometric representation theory, these are ‘intertwining maps’ between ‘categorified linear operators’. Other occurrences of spans require a more general definition of maps between spans in which, given parallel spans, a map between them consists of an object together with morphisms to each of the objects of the two spans, and invertible 2-cells making the diagram commute up to isomorphism.

## 4.2 Composition Operations on Spans

We describe the composition operations in this section using the pullback construction in Definition 1.

**Definition 13.** For each cospan diagram:



in  $B$ , we choose (and denote) a limit object,  $SR$ , projection morphisms:

$$\pi_R^S: SR \rightarrow R \quad \pi_S^R: SR \rightarrow S,$$

and an invertible 2-cell:

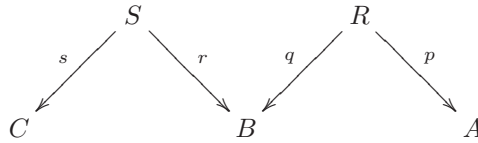
$$\kappa_B^{r,q}: q\pi_R^S \Rightarrow r\pi_S^R.$$

This data is called the **pullback of the cospan**.

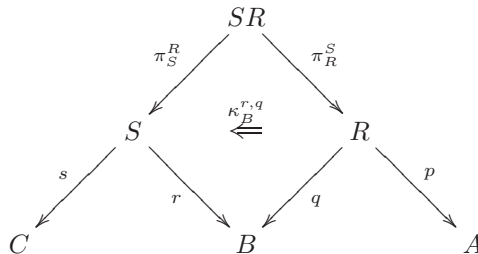
For convenience, we often refer to the object  $SR$  itself as the pullback. The chosen pullback data are fixed for the duration of the paper and the above notation will be reserved only for this data.

We think of a span as a 1-morphism from an object  $A$  to an object  $B$  in  $\text{Span}(\mathcal{B})$ .

**Definition 14.** Given a pair of composable spans:



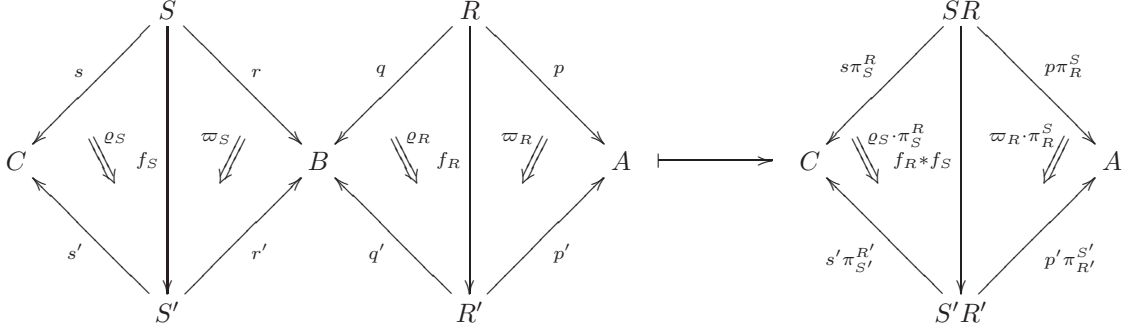
there is a **composite spans**  $SR$  from  $A$  to  $C$ :



formed by the pullback.

We use the pullback to define composition of maps of composable spans.

**Definition 15.** Given a pair of maps of spans between composable pairs of spans, we define the **horizontal composite of maps of spans**:



by the assignment:

$$((\varpi_R, f_R, \varrho_R), (\varpi_S, f_S, \varrho_S)) \mapsto (\varpi_R \cdot \pi_R^S, f_R * f_S, \varrho_S \cdot \pi_S^R),$$

where

$$f_R * f_S: SR \rightarrow S'R'$$

is the unique 1-cell satisfying:

$$\pi_{R'}^{S'}(f_R * f_S) = f_R \pi_R^S, \quad \pi_{S'}^{R'}(f_R * f_S) = f_S \pi_S^R, \quad \text{and} \quad \kappa_B^{r', q'} \cdot (f_R * f_S) = (\varpi_S \cdot \pi_S^R) \kappa_B^{r, q} (\varrho_S^{-1} \cdot \pi_R^S).$$

The pullback also allows us to define the composite of maps of maps of spans. (We omit the 2-cell components of the maps of spans  $(\varpi_S, f_S, \varrho_S)$ , etc., from the diagrams below for aesthetic purposes.) In the following definition, we need to apply the universal property of pullbacks to define the necessary 2-cell. This requires that the equation:

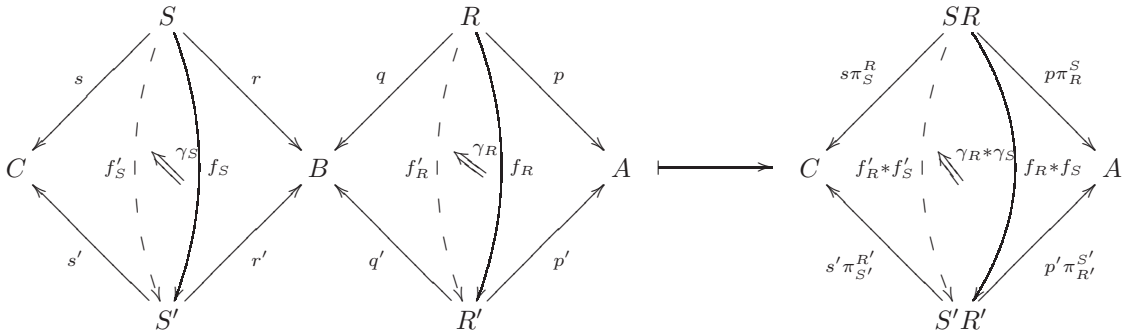
$$(r' \cdot \gamma_S \cdot \pi_S^R)(\kappa_B^{r', q'} \cdot (f_R * f_S)) = (\kappa_B^{r', q'} \cdot (f'_R * f'_S))(q' \cdot \gamma_R \cdot \pi_R^S)$$

holds, which is straightforward to verify, although we leave the details to the reader.

**Definition 16.** Given a pair of maps of maps of spans:

$$\gamma_R: (\varpi_R, f_R, \varrho_R) \Rightarrow (\varpi'_R, f'_R, \varrho'_R) \quad \text{and} \quad \gamma_S: (\varpi_S, f_S, \varrho_S) \Rightarrow (\varpi'_S, f'_S, \varrho'_S)$$

between pairs of composable maps of spans, we define the **horizontal composite of maps of maps of spans**:



by the assignment:

$$(\gamma_R, \gamma_S) \mapsto \gamma_R * \gamma_S: (\varpi_R \cdot \pi_R^S, f_R * f_S, \varrho_S \cdot \pi_S^R) \Rightarrow (\varpi'_R \cdot \pi_R^S, f'_R * f'_S, \varrho'_S \cdot \pi_S^R),$$

where  $\gamma_R * \gamma_S$  is the unique 2-cell in  $\mathcal{B}$  satisfying:

$$\pi_{R'}^{S'} \cdot (\gamma_R * \gamma_S) = \gamma_R \cdot \pi_R^S \quad \text{and} \quad \pi_{S'}^{R'} \cdot (\gamma_R * \gamma_S) = \gamma_S \cdot \pi_S^R.$$

From the defining equations of the 2-cell  $\gamma_R * \gamma_S$ , we have the equations:

$$(p' \pi_{R'}^{S'} \cdot (\gamma_R * \gamma_S))(\varpi_R \cdot \pi_R^S) = \varpi'_R \cdot \pi_R^S$$

and

$$(s' \pi_{S'}^{R'} \cdot (\gamma_R * \gamma_S))(\varrho_S \cdot \pi_S^R) = \varrho'_S \cdot \pi_S^R,$$

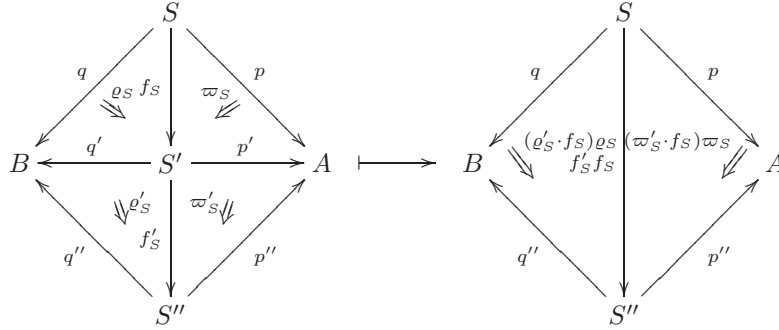
verifying that  $\gamma_R * \gamma_S$ , indeed, defines a map of maps of spans.

The three composition operations defined above were each induced by the universal property of the pullback. There is a second set of composition operations obtained via the composition operations of  $\mathcal{B}$ , which we now define.

**Definition 17.** Given a pair of composable maps of spans:

$$(\varpi_S, f_S, \varrho_S): S \rightarrow S' \text{ and } (\varpi'_S, f'_S, \varrho'_S): S' \rightarrow S''$$

between parallel spans, we define the **vertical composite of maps of spans**:



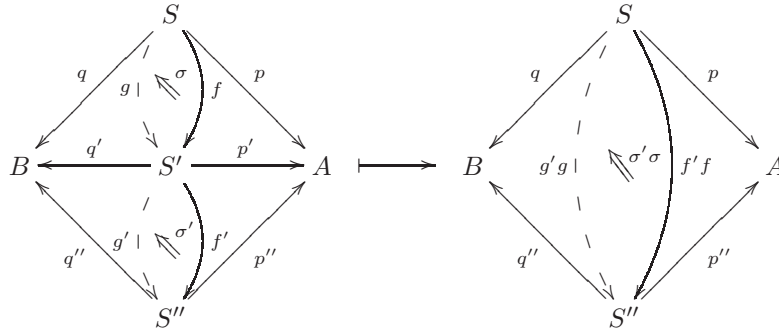
by the assignment:

$$((\varpi_S, f_S, \varrho_S), (\varpi'_S, f'_S, \varrho'_S)) \mapsto ((\varpi'_S \cdot f_S) \varpi_S, f'_S f_S, (\varrho'_S \cdot f_S) \varrho_S).$$

**Definition 18.** Given a pair of composable maps of maps of spans:

$$\sigma: (\varpi, f, \varrho) \Rightarrow (\varsigma, g, \varphi): S \rightarrow S' \text{ and } \sigma': (\varpi', f', \varrho') \Rightarrow (\varsigma', g', \varphi'): S' \rightarrow S'',$$

we define the **vertical composite of maps of maps of spans**:



by the assignment:

$$(\sigma, \sigma') \mapsto \sigma' \sigma: ((\varpi' \cdot f) \varpi, f' f, (\varrho' \cdot f) \varrho) \Rightarrow ((\varsigma' \cdot g) \varsigma, g' g, (\varphi' \cdot g) \varphi),$$

where  $\sigma' \sigma: f' f \Rightarrow g' g$  is the horizontal composite of 2-cells in  $\mathcal{B}$ . (We suppress the 2-cell components of the maps of spans for readability.)

That  $\sigma' \sigma$  is a map of maps of spans follows from the interchange law and that  $\sigma$  and  $\sigma'$  are maps of maps of spans. We have:

$$(\varsigma' \cdot f) \varsigma = ((p'' \cdot \sigma') \varpi) \cdot f((p' \cdot \sigma) \varpi) = p'' \cdot (\sigma' \sigma)((\varpi' \cdot f) \varpi)$$

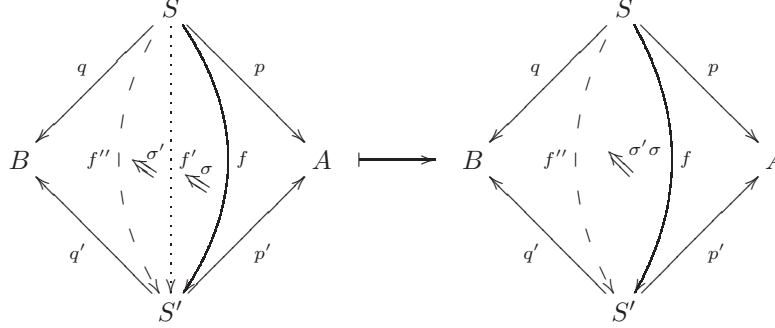
and

$$(\varphi' \cdot g) \varphi = ((q'' \cdot \sigma') \varrho) \cdot g((q' \cdot \sigma) \varrho) = q'' \cdot (\sigma' \sigma)((\varrho' \cdot g) \varrho).$$

**Definition 19.** Given a pair of composable maps:

$$\sigma: (\varpi, f, \varrho) \Rightarrow (\varpi', f', \varrho') \quad \text{and} \quad \sigma': (\varpi', f', \varrho') \Rightarrow (\varpi'', f'', \varrho'')$$

between parallel maps of spans from  $S$  to  $S'$ , we define the **horizontal composite of maps of parallel maps of spans**:



by the assignment:

$$(\sigma, \sigma') \mapsto \sigma' \sigma: (\varpi, f, \varrho) \Rightarrow (\varpi'', f'', \varrho''),$$

where  $\sigma' \sigma: f \Rightarrow f''$  is the vertical composite of 2-cells in  $\mathcal{B}$ . (We suppress the 2-cell components of the maps of spans for readability.)

It is straightforward to check that  $\sigma' \sigma$  is a map of maps of spans. We have:

$$(p' \cdot \sigma' \sigma) \varpi = (p' \cdot \sigma') (p' \cdot \sigma) \varpi = (p' \cdot \sigma') \varpi' = \varpi''$$

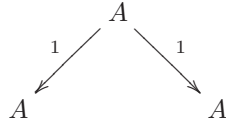
and

$$(q' \cdot \sigma' \sigma) \varrho = (q' \cdot \sigma') (q' \cdot \sigma) \varrho = (q' \cdot \sigma') \varrho' = \varrho''.$$

### 4.3 Identity Morphisms

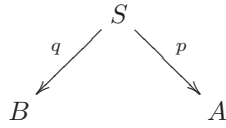
For each composition operation in the tricategory  $\text{Span}(\mathcal{B})$ , we define an identity morphism.

**Definition 20.** Given an object  $A \in \mathcal{B}$ , the **identity span** is the diagram:



This is the identity for the horizontal composition operation on spans in Definition 14.

**Definition 21.** Given a span



in  $\mathcal{B}$ , the **identity map of spans** consists of the identity 1-cell for  $S$  and the identity 2-cells  $p \Rightarrow p1_S$  and  $q \Rightarrow q1_S$  in  $\mathcal{B}$ .

This is the identity for both the horizontal and the vertical composition operations on maps of spans in Definition 15 and Definition 17, respectively. Note that  $p1_S = p$  and  $q1_S = q$ , since  $\mathcal{B}$  is a strict 2-category.

**Definition 22.** Given a map of spans  $(\varpi, f, \varrho)$ , the **identity map of maps of spans** consists of the identity 2-cell  $1_f: f \Rightarrow f$  in  $\mathcal{B}$ .

This is the identity for the horizontal and vertical composition operations on maps of maps of spans in Definition 16 and Definition 18, respectively, and the horizontal composition operation on maps of parallel spans in Definition 19.



## 4.4 Strict Hom-2-Categories

We describe the local structure of the tricategory of spans.

**Proposition 23.** *For each pair of objects  $A, B \in \mathcal{B}$ , there is a strict 2-category  $\text{Span}(\mathcal{B})(A, B)$  consisting of:*

- *spans from  $A$  to  $B$  as objects (see Definition 9),*
- *for each pair  $R, S$  of spans from  $A$  to  $B$ , a category  $\text{Span}(\mathcal{B})(A, B)(R, S)$ , consisting of:*
  - *maps of spans (see Definition 10),*
  - *maps of maps of spans (see Definition 11),*
  - *a composition operation on maps between parallel maps of spans (see Definition 19),*
  - *the identity map of maps of spans (see Definition 22),*
- *for each triple  $R, S, T$  of spans from  $A$  to  $B$ , a composition functor:*

$$*_v : \text{Span}(\mathcal{B})(A, B)(R, S) \times \text{Span}(\mathcal{B})(A, B)(S, T) \rightarrow \text{Span}(\mathcal{B})(A, B)(R, T),$$

*consisting of:*

- *a vertical composition operation on maps of spans (see Definition 17),*
- *a vertical composition operation on maps of maps of spans (see Definition 18),*
- *and, for each span  $R$  from  $A$  to  $B$ , a unit functor, consisting of the corresponding identity map of spans and identity map of maps of spans (see Definitions 21 and 22).*

*Proof.* Local composition is defined by vertical composition of 2-cells in the strict 2-category  $\mathcal{B}$ . It follows that  $\text{Span}(\mathcal{B})(A, B)(R, S)$  is a category. Functoriality of composition of maps of maps of spans follows from the interchange law for horizontal and vertical composition of 2-cells in the 2-category  $\mathcal{B}$ . Preservation of identities is immediate. The axioms are straightforward from the associative and unital composition of 1-cells and 2-cells in  $\mathcal{B}$ .  $\square$

**Remark 24.** *We will often write the vertical composites as concatenation suppressing the symbol  $*_v$ .*

## 4.5 Strict 2-Functors

We define *composition* and *unit* strict 2-functors between strict hom-2-categories.

### Composition 2-Functor

We first define horizontal composition.

**Proposition 25.** *For each triple of objects  $A, B, C \in \mathcal{B}$ , there is a strict 2-functor:*

$$*_h : \text{Span}(A, B) \times \text{Span}(B, C) \rightarrow \text{Span}(A, C),$$

*consisting of:*

- *horizontal composition of spans (see Definition 14),*
- *horizontal composition of maps of spans (see Definition 15), and*
- *horizontal composition of maps of maps of spans (see Definition 16).*

*Proof.* We need to check that horizontal composition operation on maps of maps of spans preserves vertical composition and identities, and that the naturality equations expressing functoriality for composition of maps of spans hold. The axioms of a 2-functor are immediate since  $\mathcal{B}$  is strict, i.e., the associator natural isomorphisms are identities.

Functoriality of composition of maps of maps of spans is immediate, i.e., given horizontally composable pairs of vertically composable pairs of maps of maps of spans,  $(\sigma_R, \tau_R)$  and  $(\sigma_S, \tau_S)$ , the equations:

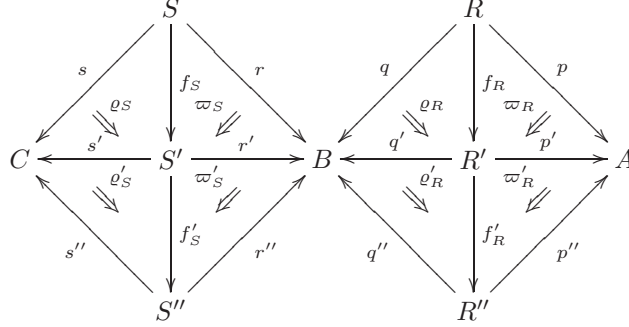
$$(\sigma_S *_v \sigma_R) *_h (\tau_S *_v \tau_R) = (\tau_R *_h \tau_S) *_v (\sigma_R *_h \sigma_S)$$

and

$$1_R *_h 1_S = 1_{RS}$$

hold.

We check that vertical composite of maps of spans is preserved by horizontal composition of maps of spans. Consider the maps of spans:



Beginning with vertical composite followed by horizontal composite, we have:

$$((\varpi'_R \cdot f_R \pi_R^S)(\varpi_R \cdot \pi_R^S), (f'_R *_v f_R) *_h (f'_S *_v f_S), (\varrho'_S \cdot f_S \pi_S^R)(\varrho_S \cdot \pi_S^R))$$

Beginning with horizontal composite followed by vertical composite, we have:

$$((\varpi'_R \cdot \pi_{R'}^{S'}(f_R *_h f_S))(\varpi_R \cdot \pi_R^S), (f'_R *_h f'_S) *_v (f_R *_h f_S), (\varrho'_S \cdot \pi_{S'}^{R'}(f_R *_h f_S))(\varrho_S \cdot \pi_S^R)).$$

These maps of spans are equal, thus composition is preserved on the nose. Similarly, identity maps of spans are preserved. It follows from these naturality equations of composites of maps of spans that the 2-functor is strict. It follows that horizontal composition is a strict 2-functor on hom-2-categories.  $\square$

**Remark 26.** We will often write the horizontal composite  $*_h$  simply as  $*$ . This should not cause confusion with the vertical composite, which will usually be written as concatenation.

## Unit 2-Functor

**Proposition 27.** For each object  $A \in \mathcal{B}$ , there is a strict 2-functor

$$I_A : \mathbf{1} \rightarrow \text{Span}(A, A),$$

which consists only of the identity span, identity map of spans, and the identity map of maps of spans (see Definitions 20, 21, and 22).

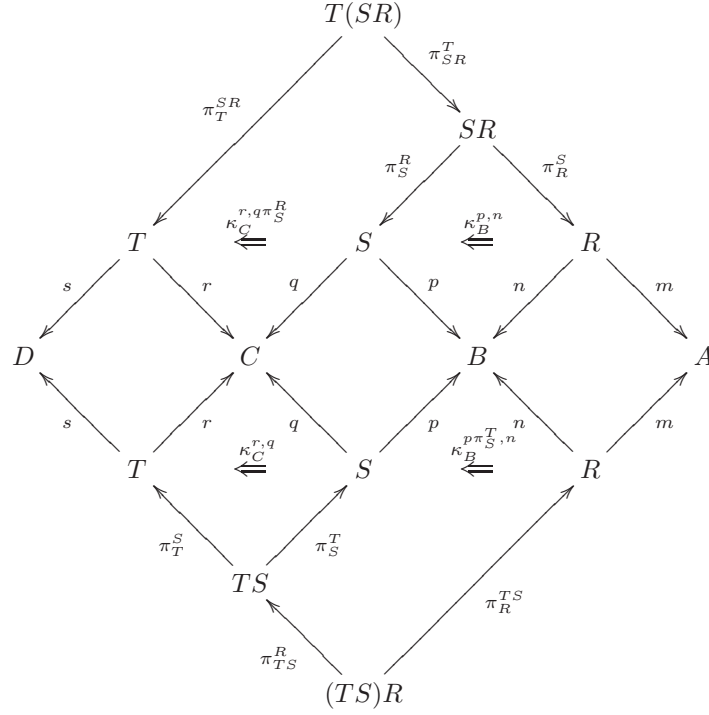
*Proof.* Straightforward.  $\square$

## 4.6 Adjoint Equivalence Transformations

We define associativity and unit adjoint equivalences in 2-categories  $[\mathcal{C}, \mathcal{C}']$  of maps between 2-categories. These 2-categories of maps are strict if the codomain 2-category  $\mathcal{C}'$  is strict. Since the hom-2-categories of the span construction are all strict, then we consider only adjoint equivalences in strict 2-categories, which simplifies the bicategorical triangle axioms of an adjoint equivalence. Each internal adjunction consists of an adjoint pair of transformations together with both counit and unit modifications. In case these modifications are trivial, then we say the adjoint equivalence is *strict*.

## Associator

To define the components of the associator transformation, first consider the diagram:



Applying the universal property induces the unique 1-cell  $a: T(SR) \rightarrow TS$  satisfying the equations:

$$\pi_S^T a = \pi_S^R \pi_{SR}^T, \quad \pi_T^S a = \pi_T^{SR}, \quad \text{and} \quad \kappa_C^{r,q} \cdot a = \kappa_C^{r,q} \pi_S^R.$$

Similarly, apply the universal property to induce the unique 1-cell  $a^{-1}: (TS)R \rightarrow SR$  satisfying the equations:

$$\pi_R^S \cdot a^{-1} = \pi_R^{TS}, \quad \pi_S^R \cdot a^{-1} = \pi_S^T \pi_{TS}^R, \quad \text{and} \quad \kappa_B^{p,n} \cdot a^{-1} = \kappa_B^{p,n} \pi_S^T.$$

Another application of the universal property yields the desired 1-cell components of the transformation and its inverse in the following proposition.

**Proposition 28.** *For each 4-tuple of objects  $A, B, C, D \in \mathcal{B}$  there is an strict adjoint equivalence  $(\mathbf{a}, \mathbf{a}^{-1}, 1, 1)$  in the strict 2-category  $[\text{Span}(A, B) \times \text{Span}(B, C) \times \text{Span}(B, D), \text{Span}(A, D)]$ , with:*

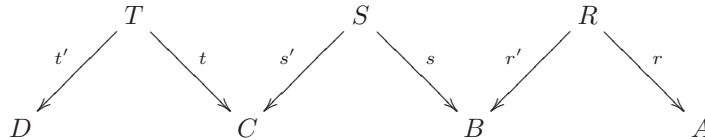
- *strict transformations*

$$\mathbf{a}: {}_{(AB)CD} (*) ({}_{ABC} * \mathbf{1}_D) \Rightarrow {}_{A(BC)D} (\mathbf{1}_A \times {}_{BCD} *),$$

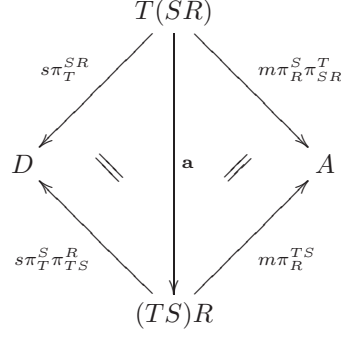
and

$$\mathbf{a}^{-1}: {}_{A(BC)D} (\mathbf{1}_A \times {}_{BCD} *) \Rightarrow {}_{(AB)CD} ({}_{ABC} * \mathbf{1}_D)$$

consisting of, for each triple of composable spans:



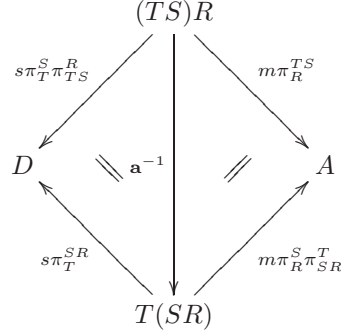
- a map of spans  $\mathbf{a}_{RST}$ :



where  $\mathbf{a}$  is a 1-cell in  $\mathcal{B}$  satisfying:

$$\pi_R^{TS} \mathbf{a} = \pi_R^S \pi_{SR}^T, \quad \pi_S^T \pi_{TS}^R \mathbf{a} = \pi_S^R \pi_{TS}^T, \quad \pi_T^{SR} \pi_{TS}^R \mathbf{a} = \pi_T^{SR}, \quad \text{and} \quad \kappa_B^{r\pi_S^T, q} \cdot \mathbf{a} = \kappa_B^{r, q} \cdot \pi_{SR}^T,$$

- and a map of spans  $\mathbf{a}_{RST}^{-1}$ :

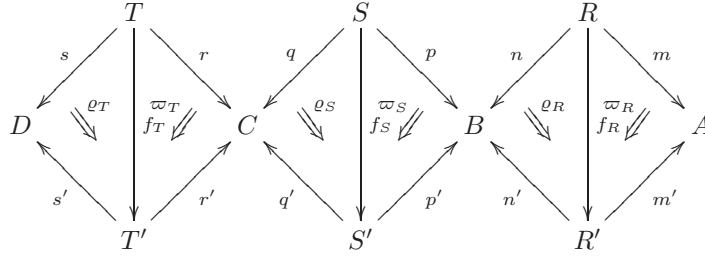


where  $\mathbf{a}^{-1}$  is a 1-cell in  $\mathcal{B}$  satisfying:

$$\pi_R^{S\pi_{SR}^T} \mathbf{a}^{-1} = \pi_R^{TS}, \quad \pi_S^R \pi_{SR}^T \mathbf{a}^{-1} = \pi_S^T \pi_{TS}^R, \quad \pi_T^{SR} \mathbf{a}^{-1} = \pi_T^S \pi_{TS}^R, \quad \text{and} \quad \kappa_C^{r, q\pi_S^R} \cdot \mathbf{a}^{-1} = \kappa_C^{r, q} \cdot \pi_{TS}^R,$$

respectively,

- for each triple of maps of composable spans:



respective naturality equations between maps of spans:

$$((\varpi_R \cdot \pi_R^S) \cdot \pi_{SR}^T, \mathbf{a}'((f_R * f_S) * f_T), \varrho_T \cdot \pi_T^{SR}) = ((\varpi_R \cdot \pi_R^{TS}) \mathbf{a}, (f_R * (f_S * f_T)) \mathbf{a}, ((\varrho_T \cdot \pi_T^S) \cdot \pi_{TS}^R) \mathbf{a})$$

and

$$(\varpi_R \cdot \pi_R^{TS}, \mathbf{a}'^{-1}(f_R * (f_S * f_T)), (\varrho_T \cdot \pi_T^S) \cdot \pi_{TS}^R) = ((\varpi_R \cdot \pi_R^S) \cdot \pi_{SR}^T \mathbf{a}^{-1}, ((f_R * f_S) * f_T) \mathbf{a}^{-1}, (\varrho_T \cdot \pi_T^{SR}) \mathbf{a}^{-1}),$$

- and identity modifications consisting of, for each span, equations:

$$(1, \mathbf{a} \mathbf{a}^{-1}, 1) = (1, 1, 1) \quad \text{and} \quad (1, 1, 1) = (1, \mathbf{a}^{-1} \mathbf{a}, 1).$$

*Proof.* The equations for  $\mathbf{a}$  and  $\mathbf{a}^{-1}$  are obtained by combining the uniqueness equations for these 1-cells with the uniqueness equations for  $a$  and  $a^{-1}$ , respectively. The transformation and modification axioms are immediate, as are the adjoint equivalence axioms. The result follows.  $\square$

## Left Unitor

We define, for each pair of objects  $A, B \in \mathcal{B}$ , the components of the left unitor adjoint equivalence  $(\mathbf{l}, \mathbf{l}^{-1}, \epsilon_{\mathbf{l}}, \eta_{\mathbf{l}})$ . The adjoint equivalence is a pair of strong transformations together with unit and counit modifications.

**Proposition 29.** *For each pair of objects  $A, B \in \mathcal{B}$  there is an adjoint equivalence:*

$$(\mathbf{l}, \mathbf{l}^{-1}, \epsilon_{\mathbf{l}}, \eta_{\mathbf{l}}): {}_{*AB1} (I_B \times 1) \Rightarrow 1,$$

in the strict 2-category  $[\text{Span}(A, B) \times \text{Span}(B, B), \text{Span}(A, B)]$ , with:

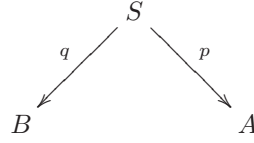
- strong transformations:

$$\mathbf{l}: {}_{*ABB} (I_B \times 1) \Rightarrow 1$$

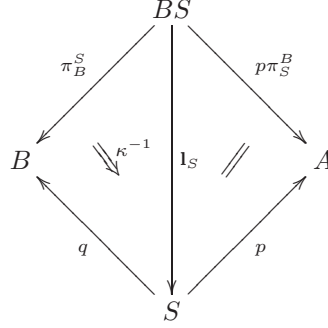
and

$$\mathbf{l}^{-1}: 1 \Rightarrow {}_{*ABB} (I_B \times 1)$$

consisting of, for each span:

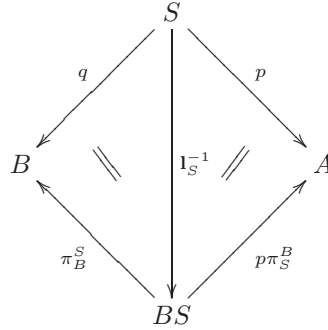


- a map of spans  $\mathbf{l}_S$ :



where  $\mathbf{l}_S := \pi_S^B$  and  $\kappa := \kappa_B^{1,q}$ , and

- a map of spans  $\mathbf{l}_S^{-1}$ :



where  $\mathbf{l}_S^{-1} := \pi_{BS}^S$ , the unique 1-cell in  $\mathcal{B}$  satisfying:

$$\pi_S^B \mathbf{l}_S^{-1} = 1, \quad \pi_B^S \mathbf{l}_S^{-1} = q, \quad \text{and} \quad \kappa_B^{1,q} \cdot \mathbf{l}_S^{-1} = 1,$$

respectively, and

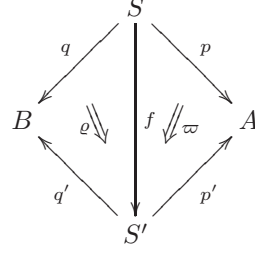
- for each pair of parallel spans  $S, S'$ , a pair of natural isomorphisms:

$$\mathbf{l}: (\mathbf{l}_{S'})_* ({}_{*}(I \times 1)) \Rightarrow \mathbf{l}_S^*$$

and

$$\mathbf{l}^{-1}: (\mathbf{l}_S^{-1})_* 1 \Rightarrow {}_{*}(I \times 1)(\mathbf{l}_{S'}^{-1})^*,$$

consisting of, for each map of spans:



– the equation of maps of spans:

$$\mathbf{l}_f : (\varpi \cdot \pi_S^B, \mathbf{l}_{S'}(f * 1), \kappa'^{-1} \cdot (f * 1)) = (\varpi \cdot \pi_S^B \mathbf{l}_S, f \mathbf{l}_S, (\varrho \cdot \pi_S^B \mathbf{l}_S) \kappa^{-1}),$$

– and the isomorphism of maps of spans:

$$\mathbf{l}_f^{-1} : (\varpi, \mathbf{l}_{S'}^{-1} f, \varrho) \Rightarrow ((\varpi \cdot \pi_S^B) \cdot \mathbf{l}_S^{-1}, (f * 1) \mathbf{l}_S^{-1}, 1),$$

defined by the unique 2-cell:

$$\mathbf{l}_f^{-1} : \mathbf{l}_{S'}^{-1} f \Rightarrow (f * 1) \mathbf{l}_S,$$

in  $\mathcal{B}$  satisfying:

$$\pi_{S'}^B \cdot \mathbf{l}_f^{-1} = 1; \text{ and } \pi_B^{S'} \cdot \mathbf{l}_f^{-1} = \varrho^{-1},$$

respectively, and,

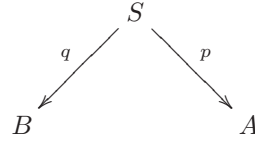
- a pair of invertible modifications:

$$\epsilon_{\mathbf{l}} : \mathbf{l}^{-1} \Rightarrow 1$$

and

$$\eta_{\mathbf{l}} : 1 \Rightarrow \mathbf{l}^{-1} \mathbf{l},$$

consisting of, for each span:



– an equation of maps of spans:

$$\epsilon_{\mathbf{l}_S} : (1, \mathbf{l}_S \mathbf{l}_S^{-1}, \kappa_B^{1,q^{-1}} \cdot \mathbf{l}_S^{-1}) = (1, 1, 1),$$

– and, an isomorphism of maps of spans:

$$\eta_{\mathbf{l}_S} : (1, 1, 1) \Rightarrow (1, \mathbf{l}_S^{-1} \mathbf{l}_S, \kappa_B^{1,q^{-1}}),$$

defined by the unique 2-cell:

$$\eta_{\mathbf{l}_S} : 1 \Rightarrow \mathbf{l}_S^{-1} \mathbf{l}_S$$

in  $\mathcal{B}$  satisfying:

$$\pi_S^B \cdot \eta_{\mathbf{l}_S} = 1 \text{ and } \pi_B^S \cdot \eta_{\mathbf{l}_S} = \kappa_B^{1,q^{-1}}.$$

*Proof.* We need to verify naturality for the components of the transformations and and verify the axioms of a transformation. It will then follow that the transformations are strong since all 2-cell data is defined via the universal property and is therefore invertible.

The equation of 2-cells:

$$(1 \cdot (\pi_B^{S'} \cdot \mathbf{l}_f^{-1}))(\kappa_B^{1,q'} \cdot \mathbf{l}_{S'}^{-1} f) = (\kappa_B^{1,q'} \cdot (f * 1_B) \mathbf{l}_S^{-1})(q' \cdot (\pi_{S'}^B \cdot \mathbf{l}_f^{-1}))$$

allows us to apply the universal property to obtain the 2-cell  $\mathbf{l}_f^{-1}$ .

For each map of maps of spans:

$$\sigma: (\varpi_f, f, \varrho_f) \Rightarrow (\varpi_g, g, \varrho_g)$$

it follows from the equation:

$$((\sigma * 1) \cdot \mathbf{l}_S^{-1}) \mathbf{l}_f^{-1} = \mathbf{l}_g^{-1} (\mathbf{l}_{S'}^{-1} \cdot \sigma)$$

that  $\mathbf{l}^{-1}$  is a natural isomorphism.

Since composition of maps of maps of spans is strictly associative and unital, and the composition and unit 2-functors of the span construction are strict, the transformation axioms reduce to the equation of maps of maps of spans:

$$\mathbf{l}_g^{-1} \mathbf{l}_f^{-1} = \mathbf{l}_{gf}^{-1}.$$

It follows that  $\mathbf{l}^{-1}$  is a strong transformation as desired. Since the natural isomorphism  $\mathbf{l}$  is the identity, the transformation  $\mathbf{l}$  is strict.

The equation of 2-cells:

$$(1 \cdot (\pi_B^S \cdot \eta_S))(\kappa_B^{1,q} \cdot 1) = (\kappa_B^{1,q} \cdot \mathbf{l}_S^{-1} \mathbf{l}_S)(q \cdot (\pi_S^B \cdot \eta_S))$$

is an equation of identity 2-cells. We can apply the universal property to define the component 2-cells of the unit modification:

$$\eta_S: (1, 1, 1) \Rightarrow (1, \mathbf{l}_S^{-1} \mathbf{l}_S, \kappa^{-1})$$

as the unique 2-cells:

$$\eta_S: 1 \Rightarrow \mathbf{l}_S^{-1} \mathbf{l}_S.$$

Finally, we have the bicategorical triangle identities. The first identity reduces to the equation of isomorphisms of maps of spans:

$$\eta_S \mathbf{l}_{\mathbf{l}_S^{-1}} = 1.$$

The second identity reduces to the equation:

$$\mathbf{l}_S \eta_S = 1.$$

Recall that the vertical composite of maps of maps of spans is defined by horizontal composition of 2-cells in  $\mathcal{B}$ . Both equations follow and we have the desired adjoint equivalence.  $\square$

## Right Unitor

We define, for each pair of objects  $A, B \in \mathcal{B}$ , the components of the right unitor adjoint equivalence  $(\mathbf{r}, \mathbf{r}^{-1}, \epsilon_{\mathbf{r}}, \eta_{\mathbf{r}})$ . As for the left unitor, this adjoint equivalence is a pair of strong transformations together with unit and counit modifications.

**Proposition 30.** *For each pair of objects  $A, B \in \mathcal{B}$  there is an adjoint equivalence:*

$$(\mathbf{r}, \mathbf{r}^{-1}, \epsilon_{\mathbf{r}}, \eta_{\mathbf{r}}): {}_{*AAB} (1 \times I_A) \Rightarrow 1,$$

in the strict 2-category  $[\text{Span}(A, A) \times \text{Span}(A, B), \text{Span}(A, B)]$ , with:

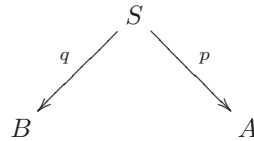
- strong transformations:

$$\mathbf{r}: {}_{*AAB} (1 \times I_A) \Rightarrow 1$$

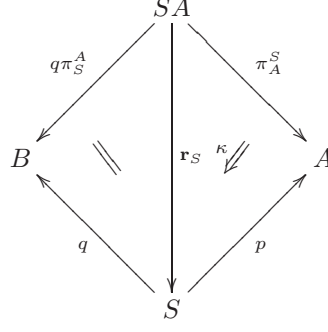
and

$$\mathbf{r}^{-1}: 1 \Rightarrow {}_{*AAB} (1 \times I_A)$$

consisting of, for each span:

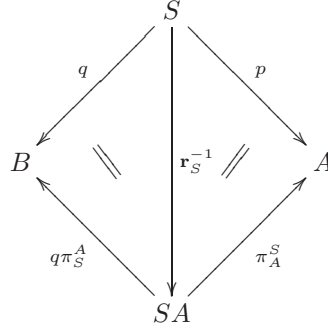


- a map of spans  $\mathbf{r}_S$ :



where  $\mathbf{r} := \pi_S^A$  and  $\kappa := \kappa_A^{p,1}$ , and

- a map of spans  $\mathbf{r}_S^{-1}$ :



where  $\mathbf{r}_S^{-1} := \pi_{SA}^S$ , the unique 1-cell in  $\mathcal{B}$  satisfying:

$$\pi_A^S \pi_{SA}^S = p \text{ and } \pi_S^A \pi_{SA}^S = 1, \text{ and } \kappa_A^{p,1} \cdot \mathbf{r}_S^{-1} = 1,$$

respectively, and

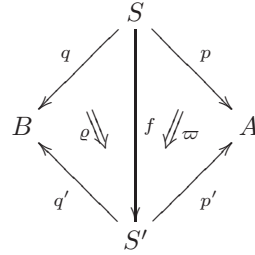
- for each pair of parallel spans  $S, S'$ , a pair of natural isomorphisms:

$$\mathbf{r}: (\mathbf{r}_{S'})_*(1 \times I) \Rightarrow 1(\mathbf{r}_S)^*$$

and

$$\mathbf{r}^{-1}: (\mathbf{r}_{S'}^{-1})_* 1 \Rightarrow *(1 \times I)(\mathbf{r}_S^{-1})^*,$$

consisting of, for each map of spans:



- the equation of maps of spans:

$$\mathbf{r}_f: (\kappa \cdot (1 * f), \mathbf{r}_{S'}(1 * f), \varrho \cdot \pi_S^A) \Rightarrow ((\varpi \cdot \mathbf{r}_S) \cdot \kappa, f \mathbf{r}_S, \varrho \cdot \mathbf{r}_S)$$

- and an isomorphism of maps of spans:

$$\mathbf{r}_f^{-1}: (\varpi, \mathbf{r}_{S'}^{-1} f, \varrho) \Rightarrow (1, (1 * f) \mathbf{r}_S^{-1}, \varrho \cdot \pi_S^A \mathbf{r}_S^{-1}),$$

defined by the unique 2-cell:

$$\mathbf{r}_f^{-1}: \mathbf{r}_{S'}^{-1} f \Rightarrow (1 * f) \mathbf{r}_S^{-1}$$

in  $\mathcal{B}$  satisfying:

$$\pi_A^{S'} \cdot \mathbf{r}_f^{-1} = \varpi \text{ and } \pi_S^A \cdot \mathbf{r}_f^{-1} = 1,$$



respectively, and,

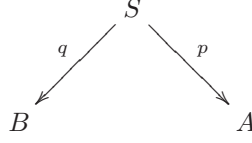
- a pair of invertible modifications:

$$\epsilon_{\mathbf{r}}: \mathbf{r}\mathbf{r}^{-1} \Rightarrow 1$$

and

$$\eta_{\mathbf{r}}: 1 \Rightarrow \mathbf{r}^{-1}\mathbf{r},$$

consisting of, for each span:



- an equation of maps of spans:

$$\epsilon_{\mathbf{r}}: (\kappa \cdot \mathbf{r}_S^{-1}, \mathbf{r}_S \mathbf{r}_S^{-1}, 1) = (1, 1, 1),$$

- and, an isomorphism of maps of spans:

$$\eta_{\mathbf{r}}: (1, 1, 1) \Rightarrow (\kappa_A^{p,1}, \mathbf{r}_S^{-1} \mathbf{r}_S, 1).$$

*Proof.* We need to verify naturality for the components of the transformations and and verify the axioms of a transformation. It will then follow that the transformations are strong since all 2-cell data is defined via the universal property and is therefore invertible.

The equation of 2-cells:

$$(p' \cdot (\pi_A^{S'} \cdot \mathbf{r}_f^{-1}))(\kappa_A^{p',1} \cdot \pi_{S'A}^{S'} f) = (\kappa_A^{p',1} \cdot (1 * f) \pi_{SA}^S)(q' \cdot (\pi_{S'}^A \cdot \mathbf{r}_f^{-1}))$$

allows us to apply the universal property to obtain the 2-cell  $\mathbf{r}_f^{-1}$ .

For each map of maps of spans:

$$\sigma: (\varpi_f, f, \varrho_f) \Rightarrow (\varpi_g, g, \varrho_g)$$

it follows from the equation:

$$(\mathbf{r}_{S'}^{-1} \cdot (1 * \sigma)) \mathbf{r}_f^{-1} = \mathbf{r}_g^{-1} (\mathbf{r}_{S'}^{-1} \cdot \sigma)$$

that  $\mathbf{r}^{-1}$  is a natural isomorphism.

Since composition of maps of maps of spans is strictly associative and unital, and the composition and unit 2-functors of the span construction are strict, the transformation axioms reduce to the equation of maps of maps of spans:

$$\mathbf{r}_g^{-1} \mathbf{r}_f^{-1} = \mathbf{r}_{gf}^{-1}.$$

It follows that  $\mathbf{r}^{-1}$  is a strong transformation as desired. Since the natural isomorphism  $\mathbf{r}$  is the identity, the transformation  $\mathbf{r}$  is strict.

The equation:

$$(p \cdot (\pi_S^A \cdot \eta_{\mathbf{r}}))(\kappa_A^{p,1} \cdot 1) = (\kappa_A^{p,1} \cdot \mathbf{r}^{-1} \mathbf{r})(1 \cdot (\pi_S^B \cdot \eta_{\mathbf{r}}))$$

is an equation of identity 2-cells. We can apply the universal property to define the component 2-cells of the unit modification:

$$\eta_{\mathbf{r}_S}: (1, 1, 1) \Rightarrow (\kappa, \mathbf{r}_S^{-1} \mathbf{r}_S, 1)$$

as the unique 2-cells:

$$\eta_{\mathbf{r}_S}: 1 \Rightarrow \mathbf{r}_S^{-1} \mathbf{r}_S.$$

Finally, we have the bicategorical triangle identities. The first identity reduces to the equation of isomorphisms of maps of spans:

$$\eta_{\mathbf{r}_S} 1_{\mathbf{r}_S^{-1}} = 1.$$

The second identity reduces to the equation:

$$1_{\mathbf{r}_S} \eta_{\mathbf{r}_S} = 1.$$

Recall that the vertical composite of maps of maps of spans is defined by horizontal composition of 2-cells in  $\mathcal{B}$ . Both equations follow and we have the desired adjoint equivalence.  $\square$

## 4.7 Invertible Modifications

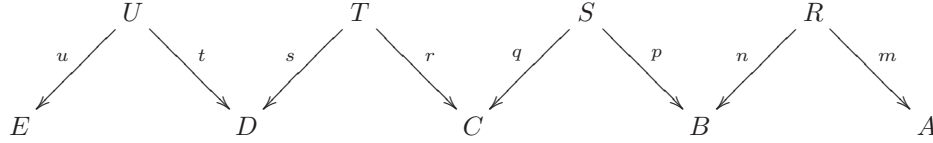
We now define four modifications each of which is an identity.

### Pentagonator Modification

**Proposition 31.** *For objects  $A, B, C, D, E \in \mathcal{B}$ , there is an identity modification:*

$$\Pi_{ABCDE}: (* \cdot (1 \times \mathbf{a}))(\mathbf{a} \cdot (1 \times * \times 1))(* \cdot (\mathbf{a} \times 1)) \Rightarrow (\mathbf{a} \cdot (1 \times 1 \times *))(* \cdot 1)(\mathbf{a} \cdot (1 \times 1 \times *)),$$

consisting of, for each four composable spans:



an equation of maps of spans:

$$(1, (1_R * \mathbf{a}_{STU})\mathbf{a}_{R(ST)U}(\mathbf{a}_{RST} * 1_U), 1) = (1, \mathbf{a}_{RS(TU)}(1_{SR} * 1_{UT})\mathbf{a}_{(RS)TU}, 1).$$

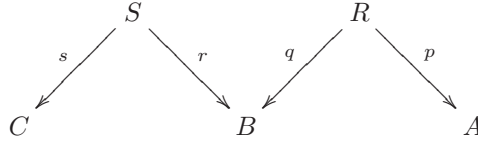
*Proof.* Straightforward. □

### Unit Modifications

**Proposition 32.** *For each triple of objects  $A, B, C \in \mathcal{B}$ , there is an identity modification:*

$$\Lambda_{ABC}: 1(* \cdot (1 \times \mathbf{1})) \Rightarrow (\mathbf{1} \cdot *) (\mathbf{a} \cdot (1 \times 1 \times I))(* \cdot 1),$$

consisting of, for each pair of composable spans:



an equation of maps of spans:

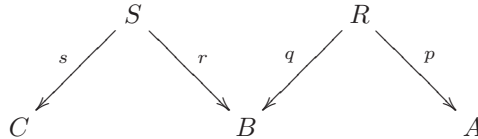
$$(1, 1_R * \mathbf{1}_S, \kappa_C^{1, s^{-1}} \cdot \pi_{CS}^R) = (1, \mathbf{1}_{SR}\mathbf{a}_{CSR}, \kappa_C^{1, s\pi_S^{R^{-1}}} \cdot \mathbf{a}_{CSR}).$$

*Proof.* Straightforward. □

**Proposition 33.** *For each triple of objects  $A, B, C \in \mathcal{B}$ , there is an identity modification:*

$$M_{ABC}: (* \cdot (\mathbf{1} \times 1)) \circ (\mathbf{a}^{-1} \cdot (1 \times I \times 1)) \circ (* \cdot (1 \times \mathbf{r}^{-1})) \Rightarrow * \cdot 1,$$

consisting of, for each pair of composable spans:



an equation of maps of spans:

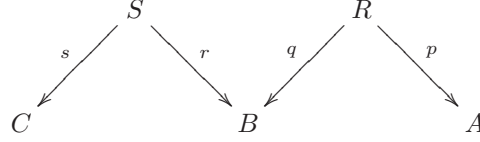
$$(1, (\mathbf{1}_R * 1_S)\mathbf{a}_{RBS}^{-1}(1_R * \mathbf{r}_S^{-1}), 1) = (1, 1_{SR}, 1).$$

*Proof.* Straightforward. □

**Proposition 34.** For each triple of objects  $A, B, C \in \mathcal{B}$ , there is an identity modification:

$$P_{ABC}: (* \cdot (1 \times \mathbf{r}))1 \Rightarrow (\mathbf{r} \cdot *)(* \cdot 1)(\mathbf{a} \cdot (I \times 1 \times 1)),$$

consisting of, for each pair of composable spans:



an identity isomorphism of maps of spans:

$$(\kappa_A^{p,1} \cdot \pi_{RA}^S, \mathbf{r}_R * 1_S, 1) = (\kappa_A^{p\pi_R^S,1} \cdot \mathbf{a}_{SRA}, \mathbf{r}_{SR}\mathbf{a}_{SRA}, 1).$$

*Proof.* Straightforward. □

## 5 Monoidal Structure on the Tricategory of Spans

In the previous section, we gave an explicit construction of the tricategory  $\text{Span}(\mathcal{B})$ , where  $\mathcal{B}$  is a strict 2-category with pullbacks. The main result of the present section is a construction of a monoidal structure on  $\text{Span}(\mathcal{B})$ , where  $\mathcal{B}$  is again a strict 2-category with pullbacks and, in addition, finite products. Recall, pullback refers to the iso-comma object and products are pseudo (or strict) products.

### 5.1 Product Operations on Spans

We define the basic components of the monoidal structure using products on the objects and morphisms of  $\text{Span}(\mathcal{B})$ .

**Definition 35.** For each pair of objects  $A, B \in \mathcal{B}$ , we choose an object denoted  $A \times B$  and projection 1-cells:

$$\tilde{\pi}_A^B: A \times B \rightarrow A \quad \text{and} \quad \tilde{\pi}_B^A: A \times B \rightarrow B$$

such that the universal property of products is satisfied.

The chosen data is called the product of  $A$  and  $B$ , however, we often refer to the object  $A \times B$  itself as the product. The above notation denoting the chosen product is fixed for the duration of the paper.

Given a pair of 1-cells  $f: A \rightarrow B$  and  $f': A' \rightarrow B'$  in  $\mathcal{B}$ , we can apply the universal property expressed in the following diagram:

$$\begin{array}{ccccc}
 A & \xleftarrow{\tilde{\pi}_A^{A'}} A \times A' & \xrightarrow{\tilde{\pi}_{A'}^A} & A' & \\
 f \downarrow & f \times f' \downarrow & & \downarrow f' & \\
 B & \xleftarrow{\tilde{\pi}_B^{B'}} B \times B' & \xrightarrow{\tilde{\pi}_{B'}^B} & B' & 
 \end{array}$$

to obtain a unique comparison 1-cell

$$f \times f': A \times A' \rightarrow B \times B'.$$

**Definition 36.** Given a pair of 1-cells  $f: A \rightarrow B$  and  $f': A' \rightarrow B'$  in  $\mathcal{B}$ , we define the **product 1-cell**

$$f \times f': A \times A' \rightarrow B \times B',$$

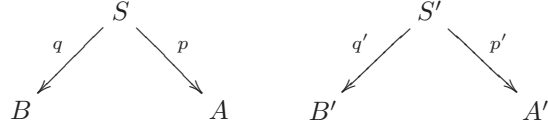
to be the unique 1-cell in  $\mathcal{B}$  such that:

$$\tilde{\pi}_B^{B'}(f \times f') = f \tilde{\pi}_A^{A'}$$

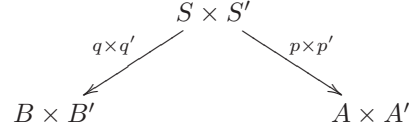
and

$$\tilde{\pi}_{B'}^B(f \times f') = f' \tilde{\pi}_{A'}^A.$$

**Definition 37.** Given a pair of spans, or 1-morphisms:



in  $\text{Span}(\mathcal{B})$ , we define the **product of spans**:



consisting of the product 1-cells  $p \times p'$  and  $q \times q'$ .

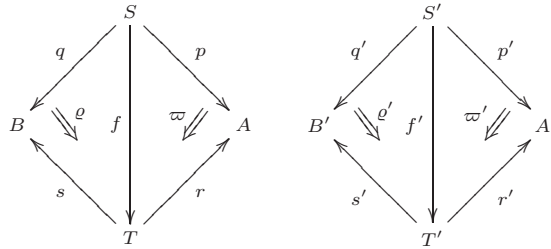
**Definition 38.** Given a pair of 2-cells  $\alpha: f \Rightarrow g$  and  $\alpha': f' \Rightarrow g'$  in  $\mathcal{B}$ , we define the **product 2-cell**:

$$\varpi \times \varpi': f \times f' \Rightarrow g \times g'$$

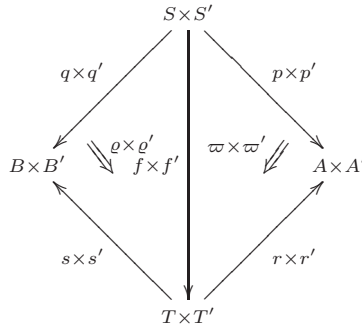
to be the unique 2-cell in  $\mathcal{B}$  satisfying:

$$\tilde{\pi}_{B'}^{B'} \cdot (\varpi \times \varpi') = \varpi \cdot \tilde{\pi}_A^{A'} \quad \text{and} \quad \tilde{\pi}_{B'}^B \cdot (\varpi \times \varpi') = \varpi' \cdot \tilde{\pi}_{A'}^A.$$

**Definition 39.** Given a pair of maps of spans, or 2-morphisms:



in  $\text{Span}(\mathcal{B})$ , we define the **map of spans**:



consisting of product 1- and 2-cells, called the **product of maps of spans**.

**Definition 40.** Given a pair of maps of maps of spans, or 3-morphisms,

$$\sigma: (\varpi, f, \varrho) \Rightarrow (\varsigma, g, \varphi) \quad \text{and} \quad \sigma': (\varpi', f', \varrho') \Rightarrow (\varsigma', g', \varphi')$$

the **product of maps of maps of spans**

$$\sigma \times \sigma': (\varpi \times \varpi', f \times f', \varrho \times \varrho') \Rightarrow (\varsigma \times \varsigma', g \times g', \varphi \times \varphi')$$

is defined by the product 2-cell

$$\sigma \times \sigma': f \times f' \Rightarrow g \times g'.$$

The following equations:

$$(r \times r') \cdot (\sigma \times \sigma')(\varpi \times \varpi') = \varsigma \times \varsigma'$$

and

$$(s \times s') \cdot (\sigma \times \sigma')(\varrho \times \varrho') = \varphi \times \varphi'$$

obtained from uniqueness in the universal property together with the 3-morphism equations for  $\varsigma$  and  $\varphi$  verify that the product of maps of maps of spans is well-defined.

## 5.2 The Monoidal Structure

**Theorem 41.** *Given a strict 2-category  $\mathcal{B}$  with pullbacks and finite products,  $\text{Span}(\mathcal{B})$  has the structure of a monoidal tricategory consisting of:*

- a locally strict trifunctor:

$$\otimes: \text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B}) \rightarrow \text{Span}(\mathcal{B})$$

defined in Proposition 42,

- a strict trifunctor

$$I: 1 \rightarrow \text{Span}(\mathcal{B}),$$

defined in Proposition 43,

- biadjoint biequivalences

- for associativity:

$$\alpha: \otimes (\otimes \times 1) \Rightarrow \otimes (1 \times \otimes),$$

in a tricategory  $[\text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B}), \text{Span}(\mathcal{B})]$  in Proposition 44,

- for left units:

$$\lambda: \otimes (I_\otimes \times 1), 1,$$

a tricategory  $[\text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B}), \text{Span}(\mathcal{B})]$  in Proposition 45,

- and, for right units:

$$\rho: \otimes (1 \times I_\otimes) \Rightarrow 1,$$

in a tricategory  $[\text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B}), \text{Span}(\mathcal{B})]$  in Proposition 46,

- adjoint equivalences

- for associativity:

$$\Pi: (1 \times \alpha)\alpha(\alpha \times 1) \Rightarrow \alpha\alpha,$$

in a bicategory  $[((\otimes \times 1)(\otimes \times 1 \times 1), \otimes \otimes (1 \times \otimes))]$ , defined in Proposition 47,

- for left units:

$$l: (1 \times \lambda)\alpha \Rightarrow \lambda,$$

in a bicategory  $[(I \times 1 \times 1)(\otimes 1) \otimes, 1 \otimes]$ , defined in Proposition 48,

- for middle units:

$$m: (1 \times \rho)\alpha \Rightarrow \lambda \times 1,$$

in a bicategory  $[1 \otimes, 1 \otimes]$ , defined in Proposition 49,

- and, for right units:

$$r: \rho\alpha \Rightarrow \rho \times 1,$$

in a bicategory  $[1 \otimes, (1 \times 1 \times I)(1 \times \otimes) \otimes]$ , defined in Proposition 50,

- invertible perturbations:

- for associativity:

$$K_5: \alpha \Pi \Pi (\Pi \times 1) \Rightarrow \Pi \alpha_{\alpha,1,1} \Pi (1 \times \Pi) \alpha_{1,\alpha,1},$$

defined in Proposition 51,

– for  $(4,1)$ -units:

$$U_{4,1}: \alpha_{\rho,1,1}r\Pi \Rightarrow (r \times 1)r\rho_{\alpha}^{-1},$$

defined in Proposition 52,

– for  $(4,2)$ -units:

$$U_{4,2}: \alpha_{\lambda,1,1}m\Pi \Rightarrow (m \times 1)\alpha_{1,\rho,1}(1 \times r),$$

defined in Proposition 53,

– for  $(4,3)$ -units:

$$U_{4,3}: m\alpha_{1,1,\rho}\Pi \Rightarrow (l \times 1)\alpha_{1,\lambda,1}(1 \times m),$$

defined in Proposition 54,

– and, for  $(4,4)$ -units:

$$U_{4,4}: l\alpha_{1,1,\rho}\Pi \Rightarrow \lambda^{-1}l(1 \times l),$$

defined in Proposition 55.

*Proof.* The structural components are given in the referenced definitions and propositions.

The monoidal structure we define involves non-trivial product cells and perturbations. This fact necessitates checking the tetracategory axioms explicitly. We verify these coherence equations in Propositions 56, 57, 58, and 59. The result follows.  $\square$

In the following sections we construct component cells of the monoidal structure on the tricategory of spans.

### 5.3 Monoidal Product

The monoidal product, which is obtained via the universal property of the product, is a locally strict trifunctor with identity modifications.

**Proposition 42.** *There is a locally strict trifunctor:*

$$\otimes: \text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B}) \rightarrow \text{Span}(\mathcal{B}),$$

consisting of:

- a function:

$$(A, B) \mapsto A \times B$$

on pairs of objects in  $\text{Span}(\mathcal{B})$  defined by the choice of strict product in Definition 2,

- for each two pairs  $(A, B), (A', B')$  objects in  $\text{Span}(\mathcal{B})$ , a strict functor

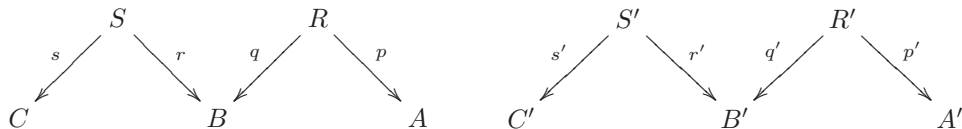
$$\otimes := \otimes_{(A,B),(A',B')}: \text{Span}(A, B) \times \text{Span}(A', B') \rightarrow \text{Span}(A \times A', B \times B')$$

between strict hom-2-categories, consisting of:

- the product of pairs of spans in Definition 37,
- the product of pairs of maps of spans in Definition 39,
- and the product of pairs of maps of maps of spans in Definition 40,
- for each two triples of objects  $(A, B, C), (A', B', C')$  in  $\text{Span}(\mathcal{B})$ , a strict adjoint equivalence:

$$(\chi, \chi^{-1}, 1, 1): * \cdot (\otimes \times \otimes) \Rightarrow \otimes(* \times *),$$

in a strict 2-category  $\text{Span}(\mathcal{B})(A \times A', C \times C')$ , consisting of, for each two pairs of composable spans:



– a map of spans:

$$\begin{array}{ccc}
 (S \times S')(R \times R') & & \\
 \begin{array}{c} \swarrow (s \times s')\pi_{S \times S'}^{R \times R'} \quad \searrow (p \times p')\pi_{R \times R'}^{S \times S'} \\ C \times C' \quad \quad \quad A \times A' \\ \nwarrow (s \times s')(\pi_S^R \times \pi_{S'}^{R'}) \quad \nearrow (p \times p')(\pi_R^S \times \pi_{R'}^{S'}) \\ SR \times S'R' \end{array} & \chi & \\
 \end{array}$$

where  $\chi := \chi_{(R,S),(R',S')}$  is a 1-cell in  $\mathcal{B}$  satisfying:

$$\begin{aligned}
 \pi_R^S \widetilde{\pi}_{SR}^{S'R'} \chi &= \widetilde{\pi}_R^{R'} \pi_{R \times R'}^{S \times S'}, & \pi_S^R \widetilde{\pi}_{SR}^{S'R'} \chi &= \widetilde{\pi}_S^{S'} \pi_{S \times S'}^{R \times R'}, \\
 \pi_{R'}^{S'} \widetilde{\pi}_{S'R'}^{SR} \chi &= \widetilde{\pi}_{R'}^R \pi_{R \times R'}^{S \times S'}, & \text{and } \pi_{S'}^{R'} \widetilde{\pi}_{S'R'}^{SR} \chi &= \widetilde{\pi}_{S'}^S \pi_{S \times S'}^{R \times R'},
 \end{aligned}$$

– and an inverse map of spans:

$$\begin{array}{ccc}
 SR \times S'R' & & \\
 \begin{array}{c} \swarrow (s \times s')(\pi_S^R \times \pi_{S'}^{R'}) \quad \searrow (p \times p')(\pi_R^S \times \pi_{R'}^{S'}) \\ C \times C' \quad \quad \quad A \times A' \\ \nwarrow (s \times s')\pi_{S \times S'}^{R \times R'} \quad \nearrow (p \times p')\pi_{R \times R'}^{S \times S'} \\ (S \times S')(R \times R') \end{array} & \chi^{-1} & \\
 \end{array}$$

where  $\chi^{-1} := \chi_{(R,S),(R',S')}^{-1}$  is a 1-cell in  $\mathcal{B}$  satisfying:

$$\pi_{R \times R'}^{S \times S'} \chi^{-1} = \pi_R^S \times \pi_{R'}^{S'}, \quad \pi_{S \times S'}^{R \times R'} \chi^{-1} = \pi_S^R \times \pi_{S'}^{R'}, \quad \text{and } \kappa_{B \times B'}^{r \times r', q \times q'} \cdot \chi^{-1} = \kappa_B^{r,q} \times \kappa_{B'}^{r',q'},$$

– identity counit and unit isomorphisms of maps of spans:

$$\epsilon_\chi: \chi \chi^{-1} \Rightarrow 1 \quad \text{and} \quad \eta_\chi: 1 \Rightarrow \chi^{-1} \chi,$$

- for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , an identity adjoint equivalence:

$$\iota_{A,B}: I_{A \times B} \Rightarrow \otimes(I_A \times I_B),$$

in the strict 2-category  $\text{Span}(\mathcal{B})(A \times B, A \times B)$ ,

- for  $(A, A'), (B, B'), (C, C'), (D, D') \in \text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B})$ , an identity modification  $\omega$  in  $\text{Span}(\mathcal{B})(A \times A', D \times D')$ , consisting of, for each pair of triples of composable spans  $(R, S, T), (R', S', T')$ , an equation of maps of spans:

$$(1, (\mathbf{a} \times \mathbf{a})\chi(\chi * 1)1) = (1, \chi(1 * \chi)\mathbf{a}, 1),$$

- for  $(A, A'), (B, B') \in \text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B})$ , an identity modification  $\gamma$  in  $\text{Span}(\mathcal{B})(A \times A', B \times B')$ , consisting of, for each pair of spans  $R, S$ , an equation of maps of spans:

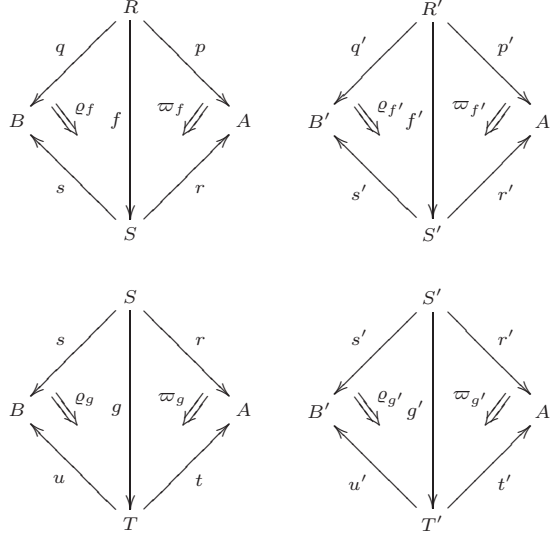
$$((\kappa^{-1} \times \kappa'^{-1}) \cdot \chi \iota, (\mathbf{1} \times \mathbf{1})\chi \iota, 1) = (\kappa^{-1}, \mathbf{1}, 1),$$

- for  $(A, A'), (B, B') \in \text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B})$ , an identity modification  $\delta$  in  $\text{Span}(\mathcal{B})(A \times A', B \times B')$ , consisting of, for each pair of spans  $R, S$ , an equation of maps of spans:

$$(1, (\mathbf{r} \times \mathbf{r})\chi \iota, (\kappa \times \kappa') \cdot \chi \iota) = (1, \mathbf{r}, \kappa).$$

*Proof.* We need to verify functoriality for maps of spans, i.e., that the 2-functor is strict, and then functoriality for the 2-functor, which concerns maps of maps of spans. We define unique auxiliary 1-cells to give unique definitions of  $\chi$  and  $\chi^{-1}$  by the universal property, and then verify naturality for the two families of maps of spans. It is then straightforward to see that these natural transformations together with identity counit and unit define a strict adjoint equivalence. It is again straightforward to see that  $\iota$  uniquely defines a strict adjoint equivalence. We then have identity modifications and together with the identity modifications of  $\text{Span}(\mathcal{B})$ , the trifunctor axioms follow.

For each two pairs of objects  $(A, B), (A', B') \in \text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B})$ , the monoidal product should preserve composition of maps of spans and identity maps of spans. It is straightforward from definitions to see that identities are preserved. For composition, consider the two pairs of maps of spans:

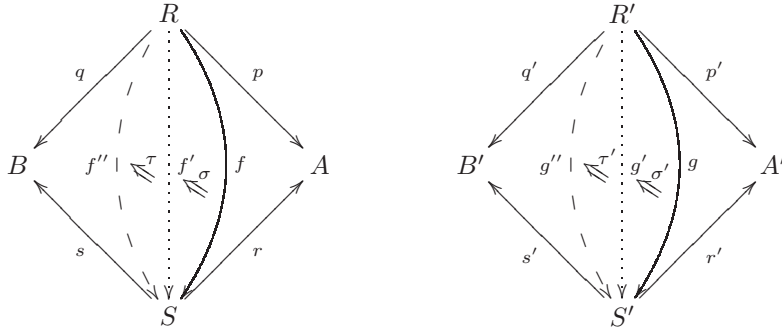


Functoriality of maps of spans follows from the equation:

$$(((\varpi_g \times \varpi_{g'}) \cdot (f \times f'))(\varpi_f \times \varpi_{f'}), (g \times g')(f \times f'), ((\varrho_g \times \varrho_{g'}) \cdot (g \times g'))(\varrho_f \times \varrho_{f'})) = ((\varpi_g \cdot f)\varpi_f \times (\varpi_{g'} \cdot f')\varpi_{f'}, gf \times g'f', (\varrho_g \cdot f)\varrho_f \times (\varrho_{g'} \cdot f')\varrho_{f'}),$$

and the obvious preservation of identity maps of spans, so the monoidal product is strict.

To verify functoriality, consider pairs of composable pairs of maps of maps of spans:



We have the equation of 2-cells

$$\tau\sigma \times \tau'\sigma' = (\tau \times \tau')(\sigma \times \sigma')$$

in  $\mathcal{B}$ , so composition is preserved. It is straightforward to see that identity maps of maps of spans are preserved. It follows that products in  $\mathcal{B}$  define a strict functor on hom-2-categories.

We define the components of a strict adjoint equivalence  $\chi$ . Recall that strict transformations are natural transformations consisting of 1-morphisms. For each pair of triples of objects  $(A, B, C), (A', B', C') \in \text{Span}(\mathcal{B}) \times \text{Span}(\mathcal{B})$ , we define a natural transformation:

$$\chi_{(A,B,C),(A',B',C')} : *(\otimes \times \otimes) \Rightarrow \otimes(* \times *).$$



We first apply the universal property of pullbacks to obtain an auxiliary pair of maps  $\chi_{SR}$  and  $\chi_{S'R'}$ , which we then use to obtain the 1-cell  $\chi$ . The 1-cells  $\chi_{SR}$  and  $\chi_{S'R'}$  are the unique 1-cells in  $\mathcal{B}$  making the diagrams:

$$\begin{array}{ccccc} S \times S' & \xleftarrow{\pi_{S \times S'}^{R \times R'}} (S \times S')(R \times R') & \xrightarrow{\pi_{R \times R'}^{S \times S'}} R \times R' \\ \tilde{\pi}_{S'}^{S'} \downarrow & \chi_{SR} \downarrow & \downarrow \tilde{\pi}_R^{R'} \\ S & \xleftarrow{\pi_S^R} SR & \xrightarrow{\pi_R^S} R \end{array}$$

$$\begin{array}{ccccc} S \times S' & \xleftarrow{\pi_{S \times S'}^{R \times R'}} (S \times S')(R \times R') & \xrightarrow{\pi_{R \times R'}^{S \times S'}} R \times R' \\ \tilde{\pi}_{S'}^{S'} \downarrow & \chi_{S'R'} \downarrow & \downarrow \tilde{\pi}_{R'}^R \\ S' & \xleftarrow{\pi_{S'}^{R'}} S'R' & \xrightarrow{\pi_{R'}^S} R' \end{array}$$

commute and satisfying the equations:

$$\kappa_B^{SR} \cdot \chi_{SR} = \tilde{\pi}_B^{B'} \cdot \kappa_{B \times B'}^{(S \times S')(R \times R')} \quad \text{and} \quad \kappa_{B'}^{S'R'} \cdot \chi_{S'R'} = \tilde{\pi}_{B'}^B \cdot \kappa_{B \times B'}^{(S \times S')(R \times R')}.$$

Applying the universal property of products, we have the unique 1-cell

$$\chi_{(R,S),(R',S')} : (S \times S')(R \times R') \rightarrow SR \times S'R'$$

such that

$$\tilde{\pi}_{SR}^{S'R'} \chi = \chi_{SR} \quad \text{and} \quad \tilde{\pi}_{S'R'}^{SR} \chi = \chi_{S'R'}.$$

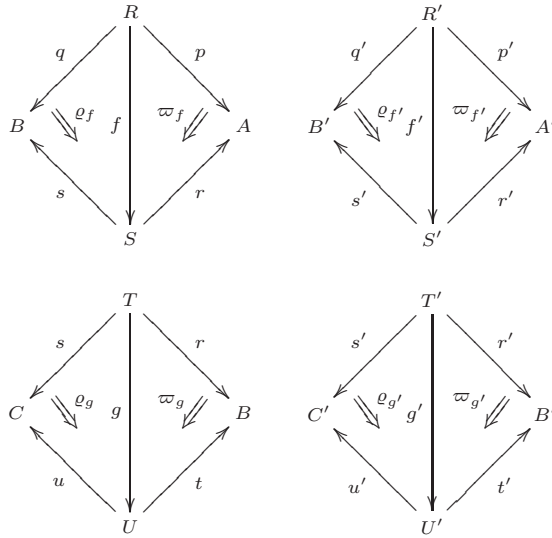
From the above equations we have:

$$\pi_R^S \tilde{\pi}_{SR}^{S'R'} \chi = \tilde{\pi}_R^{R'} \pi_{R \times R'}^{S \times S'}, \quad \pi_S^R \tilde{\pi}_{SR}^{S'R'} \chi = \tilde{\pi}_S^{S'} \pi_{S \times S'}^{R \times R'}, \quad \pi_{R'}^S \tilde{\pi}_{S'R'}^{SR} \chi = \tilde{\pi}_{R'}^R \pi_{R \times R'}^{S \times S'}, \quad \text{and} \quad \pi_{S'}^R \tilde{\pi}_{S'R'}^{SR} \chi = \tilde{\pi}_{S'}^S \pi_{S \times S'}^{R \times R'}.$$

Similarly, we define 1-cells  $\chi^{-1}$  such that:

$$\pi_{R \times R'}^{S \times S'} \chi^{-1} = \pi_R^S \times \pi_{R'}^S, \quad \pi_{S \times S'}^{R \times R'} \chi^{-1} = \pi_S^R \times \pi_{S'}^R, \quad \text{and} \quad \kappa_{B \times B'}^{(S \times S')(R \times R')} \cdot \chi^{-1} = \kappa_B^{SR} \times \kappa_{B'}^{S'R'}.$$

To verify naturality we see that, for each two pairs of horizontally composable maps of spans:



there is an identity isomorphism:

$$\begin{aligned} & ((\varpi_f \times \varpi_{f'}) \cdot \pi_{R \times R'}^{S \times S'}, \chi'((f \times f') * (g \times g')), (\varrho_g \times \varrho_{g'}) \cdot \pi_{S \times S'}^{R \times R'}) = \\ & (((\varpi_f \cdot \pi_R^S) \times (\varpi_{f'} \cdot \pi_{R'}^S))\chi, ((f * g) \times (f' * g'))\chi, ((\varrho_g \cdot \pi_S^R) \times (\varrho_{g'} \cdot \pi_{S'}^R))\chi) \end{aligned}$$

between the maps of spans:

$$\begin{array}{ccccc}
 & (S \times S')(R \times R') & & & \\
 (q \times q')\pi_{S \times S'}^{R \times R'} \swarrow & & \searrow & (m \times m')\pi_{R \times R'}^{S \times S'} & \\
 C \times C' & (\times \cdot \circ)\chi & \chi'(\circ \cdot \times) & A \times A' & \\
 (u \times u')(\pi_U^T \times \pi_{U'}^{T'}) \swarrow & & \searrow & (r \times r')(\pi_T^U \times \pi_{T'}^{U'}) & \\
 & UT \times U'T' & & &
 \end{array}$$

It follows that the maps of spans are the components of a natural transformation:

$$\chi: *(\otimes \times \otimes) \Rightarrow \otimes(* \times *).$$

One can check similarly that the collection of maps:

$$\chi^{-1}: \otimes(* \times *) \Rightarrow *(\otimes \times \otimes)$$

is a natural transformation.

The equations of maps of spans:

$$(1, \chi\chi^{-1}, 1) = (1, 1, 1) \text{ and } (1, \chi^{-1}\chi, 1) = (1, 1, 1),$$

which are the components of the identity counit and unit verify that  $(\chi, \chi^{-1}, 1, 1)$  is a strict adjoint equivalence. The axioms are immediate.

The existence of the identity adjoint equivalence  $\iota$ , and the identity modifications is straightforward. All axioms are immediate. We have the desired locally strict trifunctor.  $\square$

## 5.4 Monoidal Unit

**Proposition 43.** *There is a strict functor between tricategories called the monoidal unit, consisting of the terminal object (or nullary product) in  $B$ , and the identity morphisms of Definition 20, Definition 21, and Definition 22 for spans, maps of spans, and maps of maps of spans, respectively.*

*Proof.* Straightforward.  $\square$

## 5.5 Monoidal Biadjoint Biequivalences

The monoidal associativity and unit structure is given as biadjoint biequivalences. Note that Trimble's definition of tetracategory asks that the tritransformations be equivalences in the appropriate sense at each level. He calls such a map a triequivalence, which probably should not be interpreted as a trifunctor that is an equivalence, but rather to a suitable notion of 'strong tritransformation'. Following the definition of Gurski's algebraic tricategory, we replace these structural tritransformations with biadjoint biequivalences, a notion which categorifies that of adjoint equivalence. See Gurski's thesis [11] for definitions.

The following biadjoint biequivalences are pairs of biadjoint 1-cells in certain tricategories of trifunctors, tritransformations, trimodifications, and perturbations. This notion of biadjoint is not to be confused with ambidextrous adjoint pairs, which are sometimes called biadjoints.

Tritransformations, trimodifications, and perturbations are the morphisms of the local tricategories of a tetracategory  $\text{Tricat}$  of tricategories. We need to specify the structure of these tricategories. Gurski shows that for  $\mathcal{T}, \mathcal{T}'$  tricategories such that  $\mathcal{T}'$  is also a Gray-category, then  $\text{Tricat}(\mathcal{T}, \mathcal{T}')$  is also a Gray-category. [11] Unfortunately,  $\text{Span}(\mathcal{B})$  is not quite a Gray-category, so there is still work to do in specifying the structure of  $\text{Tricat}(\text{Span}(\mathcal{B})^n, \text{Span}(\mathcal{B}))$ , for  $n$  a natural number. As a corollary of the local bicategory construction in the local tricategories of  $\text{Tricat}$ , Gurski further shows that if  $\mathcal{T}'$  is locally strict, then for trifunctors  $F, G: \mathcal{T} \rightarrow \mathcal{T}'$ , the bicategory  $\text{Tricat}(\mathcal{T}, \mathcal{T}')(F, G)$  is a strict 2-category [11]. Since  $\text{Span}(\mathcal{B})$  is locally strict, so is  $\text{Tricat}(\text{Span}(\mathcal{B})^n, \text{Span}(\mathcal{B}))$ .

To the best of our knowledge the tricategorical structure of  $\text{Tricat}(\mathcal{T}, \mathcal{T}')$  does not exist in the literature for an arbitrary tricategory  $\mathcal{T}'$ . We expect the details should be straightforward, but we do not have space here to present the details. This is not too troublesome since we can invoke tricategorical coherence. The tricategory

$\text{Span}(\mathcal{B})$  is semi-strict and cubical. If the associator and unit tritransformations were identities, then we could apply the theorem above and use the Gray-category structure on  $\text{Tricat}(\mathcal{T}, \mathcal{T}')$ . Alternatively, we could consider a strictification of the span tricategory  $\text{Span}(\mathcal{B})$  to a triequivalent Gray-category  $\text{Span}^{\text{Gray}}(\mathcal{B})$ :

$$\text{st}: \text{Span}(\mathcal{B}) \rightarrow \text{Span}^{\text{Gray}}(\mathcal{B}).$$

We can then essentially ‘whisker’ the biadjoint biequivalence structures we define with the stratification maps. The result being a biadjoint biequivalence in the Gray-category  $[\text{Span}(\mathcal{B})^n, \text{Span}^{\text{Gray}}(\mathcal{B})]$ , rather than the locally strict tricategory  $[\text{Span}(\mathcal{B})^n, \text{Span}(\mathcal{B})]$ . This does not really provide satisfactory resolution to the issue, but instead strongly suggests that there should be numerous solutions to the problem, each of which should be expected to be relatively straightforward. We do not comment further on the tricategory structures in which we define the biadjoint biequivalences, but instead acknowledge this as a missing piece of the construction, which is not likely to trouble the reader to a large extent.

We will need the unit tritransformation:

$$I_{\otimes(\otimes \times 1)}: \otimes(\otimes \times 1) \Rightarrow \otimes(\otimes \times 1)$$

which consists of:

- for each triple of objects  $A, B, C \in \text{Span}(\mathcal{B})$ , a span  $I_{ABC}$ :

$$\begin{array}{ccc} & (A \times B) \times C & \\ 1 \swarrow & & \searrow 1 \\ (A \times B) \times C & & (A \times B) \times C \end{array}$$

and for each pair of triples of objects  $(A, B, C), (A', B', C')$ , an adjoint equivalence:

$$(I, I', \epsilon_I, \eta_I): (I_{A'B'C'})_* \otimes (\otimes \times 1) \Rightarrow (I_{ABC})^* \otimes (\otimes \times 1),$$

consisting of:

- a transformation:

$$I_{(ABC), (A'B'C')} : (I_{A'B'C'})_* \otimes (\otimes \times 1) \Rightarrow (I_{ABC})^* \otimes (\otimes \times 1),$$

consisting of:

- \* for each triple of spans  $(R, S, T)$ , a map of spans  $I_{RST}$ :

$$\begin{array}{ccccc} & ((A' \times B') \times C')((R \times S) \times T) & & & \\ \pi_{(A' \times B') \times C'}^{(R \times S) \times T} \swarrow & \downarrow I & \searrow \pi_{(R \times S) \times T}^{(A \times B) \times C} & & \\ (A' \times B') \times C' & & (A \times B) \times C & & \\ \pi_{(R \times S) \times T}^{(A \times B) \times C} \swarrow & \downarrow I & \searrow \pi_{(A \times B) \times C}^{(R \times S) \times T} & & \\ & ((R \times S) \times T)((A \times B) \times C) & & & \end{array}$$

where  $I := I_{RST}$  is the unique 1-cell in  $\mathcal{B}$  such that:

$$\pi_{(A \times B) \times C}^{(R \times S) \times T} I_{RST} = ((r \times s) \times t) \pi_{(R \times S) \times T}^{(A' \times B') \times C'}, \quad \pi_{(R \times S) \times T}^{(A \times B) \times C} I_{RST} = \pi_{(R \times S) \times T}^{(A' \times B') \times C'}$$

and

$$\kappa_{(A \times B) \times C}^{(r \times s) \times t, 1} \cdot I_{RST} = 1,$$

- \* for each pair of triples of spans  $(R, S, T), (\bar{R}, \bar{S}, \bar{T})$ , a natural isomorphism:

$$I_{(RST), (\bar{R}, \bar{S}, \bar{T})}: (I_{R, S, T})^* (I_{ABC})^* \otimes (\otimes \times 1) \Rightarrow (I_{\bar{R}, \bar{S}, \bar{T}})_* (I_{A'B'C'})_* \otimes (\otimes \times 1),$$

consisting of, for each triple of maps of spans  $(f_R, f_S, f_T)$ , an isomorphism of maps of spans:

$$I_{f_R f_S f_T} : (((\varpi_R \times \varpi_S) \times \varpi_T) \cdot \pi_{(R \times S) \times T}^{(A' \times B') \times C'}, I_{\bar{R}, \bar{S}, \bar{T}}(((f_R \times f_S) \times f_T) * 1_{(A' \times B') \times C'}), 1) \\ \Rightarrow (1, (1_{(A \times B) \times C} * ((f_R \times f_S) \times f_T)) I_{R, S, T}, ((\varrho_R \times \varrho_S) \times \varrho_T) \cdot \pi_{(R \times S) \times T}^{(A \times B) \times C} I_{R, S, T})$$

consisting of the unique 2-cell:

$$I_{f_R f_S f_T} : I_{\bar{R}, \bar{S}, \bar{T}}(((f_R \times f_S) \times f_T) * 1_{(A' \times B') \times C'}) \Rightarrow (1_{(A \times B) \times C} * ((f_R \times f_S) \times f_T)) I_{R, S, T}$$

in  $\mathcal{B}$ , such that:

$$\pi_{(A \times B) \times C}^{(\bar{R} \times \bar{S}) \times \bar{T}} \cdot I_{f_R f_S f_T} = \varpi_{(R, S) T}^{-1} \cdot \pi_{(R \times S) \times T}^{(A \times B) \times C}$$

and

$$\pi_{(\bar{R} \times \bar{S}) \times \bar{T}}^{(A \times B) \times C} \cdot I_{f_R f_S f_T} = 1,$$

– a transformation:

$$I_{(ABC), (A' B' C')} : (I_{ABC})^* \otimes (\otimes \times 1) \Rightarrow (I_{A' B' C'})_* \otimes (\otimes \times 1),$$

consisting of:

\* for each triple of spans  $(R, S, T)$ , a map of spans  $I_{RST}$ :

$$\begin{array}{ccc} & ((R \times S) \times T)((A \times B) \times C) & \\ \swarrow \pi_{(R \times S) \times T}^{(A' \times B') \times C'} \cdot ((r' \times s') \times t') & \downarrow I & \searrow \pi_{(A \times B) \times C}^{(R \times S) \times T} \\ (A' \times B') \times C' & & (A \times B) \times C \\ \nwarrow \pi_{(A' \times B') \times C'}^{(R \times S) \times T} & & \nearrow ((r \times s) \times t) \pi_{(R \times S) \times T}^{(A \times B) \times C} \\ & ((A' \times B') \times C')((R \times S) \times T) & \end{array}$$

where  $I := I_{RST}$  is the unique 1-cell in  $\mathcal{B}$  such that:

$$\pi_{(R \times S) \times T}^{(A' \times B') \times C'} I_{RST} = \pi_{(R \times S) \times T}^{(A \times B) \times C}, \quad \pi_{(A' \times B') \times C'}^{(R \times S) \times T} I_{RST} = ((r' \times s') \times t') \pi_{(R \times S) \times T}^{(A \times B) \times C}$$

and

$$\kappa_{(A' \times B') \times C'}^{1, (r' \times s') \times t'} \cdot I_{RST} = 1,$$

\* for each pair of triples of spans  $(R, S, T), (\bar{R}, \bar{S}, \bar{T})$ , a natural isomorphism:

$$I_{(RST), (\bar{R}, \bar{S}, \bar{T})} : (I_{R, S, T})^* (I_{ABC})^* \otimes (\otimes \times 1) \Rightarrow (I_{\bar{R}, \bar{S}, \bar{T}})_* (I_{A' B' C'})_* \otimes (\otimes \times 1),$$

consisting of, for each triple of maps of spans  $(f_R, f_S, f_T)$ , an isomorphism of maps of spans:

$$I_{f_R f_S f_T} : (((\varpi_R \times \varpi_S) \times \varpi_T) \cdot \pi_{(R \times S) \times T}^{(A' \times B') \times C'}, I_{\bar{R}, \bar{S}, \bar{T}}(((f_R \times f_S) \times f_T) * 1_{(A' \times B') \times C'}), 1) \\ \Rightarrow (1, (1_{(A \times B) \times C} * ((f_R \times f_S) \times f_T)) I_{R, S, T}, ((\varrho_R \times \varrho_S) \times \varrho_T) \cdot \pi_{(R \times S) \times T}^{(A \times B) \times C} I_{R, S, T})$$

consisting of the unique 2-cell:

$$I_{f_R f_S f_T} : I_{\bar{R}, \bar{S}, \bar{T}}(((f_R \times f_S) \times f_T) * 1_{(A' \times B') \times C'}) \Rightarrow (1_{(A \times B) \times C} * ((f_R \times f_S) \times f_T)) I_{R, S, T}$$

in  $\mathcal{B}$ , such that:

$$\pi_{(\bar{R} \times \bar{S}) \times \bar{T}}^{(A \times B) \times C} \cdot I_{f_R f_S f_T} = 1$$

and

$$\pi_{(A \times B) \times C}^{(\bar{R} \times \bar{S}) \times \bar{T}} \cdot I_{f_R f_S f_T} = \varrho_{(f_R f_S) f_T}^{-1} \cdot \pi_{(R \times S) \times T}^{(A \times B) \times C},$$

– a modification:

$$\epsilon_I: II \Rightarrow 1,$$

consisting of, for each triple of spans  $(R, S, T)$ , an isomorphism of maps of spans:

$$\epsilon_{IRST}: (1, I_{RST}I_{RST}, 1) \Rightarrow (1, 1_{((RS)T)((AB)C)}, 1),$$

consisting of the unique 2-cell:

$$\epsilon_{IRST}: I_{RST}I_{RST} \Rightarrow 1_{((RS)T)((AB)C)}$$

in  $\mathcal{B}$  such that:

$$\pi_{(A \times B) \times C}^{(R \times S) \times T} \cdot \epsilon_{IRST} = \kappa_{(A \times B) \times C}^{(r \times s) \times t, 1}^{-1}$$

and

$$\pi_{(R \times S) \times T}^{(A \times B) \times C} \cdot \epsilon_{IRST} = 1,$$

– a modification:

$$\eta_I: 1 \Rightarrow I \cdot I,$$

consisting of, for each triple of spans  $(R, S, T)$ , an isomorphism of maps of spans:

$$\eta_{IRST}: (1, 1_{((A'B')C')((RS)T)}, 1) \Rightarrow (1, I_{RST}I_{RST}, 1),$$

consisting of the unique 2-cell:

$$\eta_{IRST}: 1_{((A'B')C')((RS)T)} \Rightarrow I_{RST}I_{RST}$$

in  $\mathcal{B}$  such that:

$$\pi_{(R \times S) \times T}^{(A' \times B') \times C'} \cdot \eta_{IRST} = 1$$

and

$$\pi_{(A' \times B') \times C'}^{(R \times S) \times T} \cdot \eta_{IRST} = \kappa_{(A' \times B') \times C'}^{1, (r \times s) \times t},$$

• an identity modification:

$$I_{\Pi}: (1, (\mathbf{a} \times \mathbf{a})\chi(\chi * 1), 1) \Rightarrow (1, \chi(1 * \chi)\mathbf{a}, 1),$$

• and an identity modification:

$$I_M: (1, (1 * I)(\iota * 1)\mathbf{r}^{-1}, 1) \Rightarrow (1, \iota\mathbf{l}^{-1}, 1).$$

The other unit tritransformations are defined similarly.

## Monoidal Associativity

Monoidal associativity is a biadjoint biequivalence consisting of tritransformations, adjoint equivalences of trimodifications and perturbations, and coherence perturbations consisting of isomorphisms of maps of spans.

The associator for the product of objects  $A, B, C \in \mathcal{B}$  is the 1-cell:

$$a_{ABC}: (A \times B) \times C \rightarrow A \times (B \times C)$$

in  $\mathcal{B}$  defined as the product of 1-cells:

$$a_{ABC} := \tilde{\pi}_A^B \times 1_{B \times C}.$$

The inverse associator is the 1-cell:

$$a_{ABC}^{-1}: A \times (B \times C) \rightarrow (A \times B) \times C$$

in  $\mathcal{B}$  defined as the product of 1-cells:

$$a_{ABC}^{-1} := 1_{A \times B} \times \tilde{\pi}_C^B.$$

**Proposition 44.** *There is a biadjoint biequivalence:*

$$(\alpha, \alpha', \epsilon_\alpha, \eta_\alpha, \Phi_\alpha, \Psi_\alpha): \otimes (\otimes \times 1) \Rightarrow \otimes (1 \times \otimes)$$

in a ‘tricategory’  $\text{Tricat}(\text{Span}(\mathcal{B})^3, \text{Span}(\mathcal{B}))$ , consisting of:

- a tritransformation:

$$\alpha: \otimes (\otimes \times 1) \Rightarrow \otimes (1 \times \otimes),$$

consisting of:

- for each triple  $A, B, C$  of objects in  $\text{Span}(\mathcal{B})$ , a span  $\alpha_{ABC}$ :

$$\begin{array}{ccc} & (A \times B) \times C & \\ a \swarrow & & \searrow 1 \\ A \times (B \times C) & & (A \times B) \times C \end{array}$$

- for each two triples  $(A, B, C), (A', B', C')$  of objects in  $\text{Span}(\mathcal{B})$ , an adjoint equivalence:

$$(\alpha_\mu, \alpha_\mu^*, \alpha_\epsilon, \alpha_\eta): \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'}) (\otimes (\otimes \times 1)) \Rightarrow \text{Span}(\mathcal{B})(\alpha_{ABC}, 1) (\otimes (1 \times \otimes)),$$

in the strict 2-category:

$$\text{Bicat}(\text{Span}(\mathcal{B})^3((A, B, C), (A', B', C')), \text{Span}(\mathcal{B})((A \times B) \times C, A' \times (B' \times C')),$$

consisting of:

- \* a strong transformation:

$$\alpha_{\mu(A, B, C), (A', B', C')}: \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'}) (\otimes (\otimes \times 1)) \Rightarrow \text{Span}(\mathcal{B})(\alpha_{ABC}, 1) (\otimes (1 \times \otimes)),$$

consisting of:

- for each triple of spans:

$$\begin{array}{ccc} & R & \\ r' \swarrow & & \searrow r \\ A' & & A \end{array} \quad \begin{array}{ccc} & S & \\ s' \swarrow & & \searrow s \\ B' & & B \end{array} \quad \begin{array}{ccc} & T & \\ t' \swarrow & & \searrow t \\ C' & & C \end{array}$$

a map of spans:

$$\begin{array}{ccc} & ((A' \times B') \times C') ((R \times S) \times T) & \\ \alpha\pi_{(A' \times B') \times C'}^{(R \times S) \times T} \swarrow & \downarrow & \searrow ((r \times s) \times t)\pi_{(R \times S) \times T}^{(A' \times B') \times C'} \\ A' \times (B' \times C') & \xrightarrow{\alpha_\mu} & (A \times B) \times C \\ \downarrow (r' \times (s' \times t'))\pi_{R \times (S \times T)}^{(A \times B) \times C} & & \uparrow \pi_{(A \times B) \times C}^{R \times (S \times T)} \\ & (R \times (S \times T)) ((A \times B) \times C) & \end{array}$$

where  $\kappa := \kappa_{(A' \times B') \times C'}^{1, (r' \times s') \times t'}$  and  $\alpha_\mu := \alpha_{\mu RST}$  is the unique 1-cell satisfying:

$$\pi_{(A \times B) \times C}^{R \times (S \times T)} \alpha_{\mu RST} = ((r \times s) \times t)\pi_{(R \times S) \times T}^{(A' \times B') \times C'} \pi_{R \times (S \times T)}^{(A \times B) \times C} \alpha_{\mu RST} = a\pi_{(R \times S) \times T}^{(A' \times B') \times C'},$$

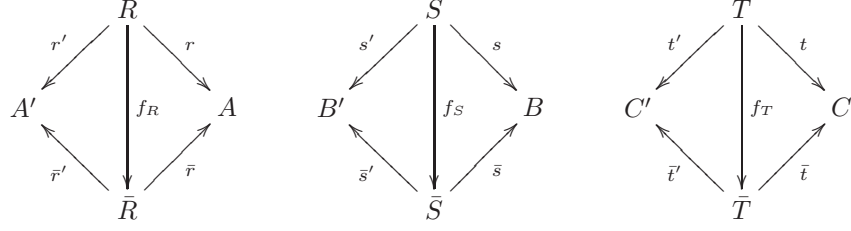
and

$$\kappa_{A \times (B \times C)}^{r \times (s \times t), 1} \cdot \alpha_{\mu RST} = 1,$$

- for each pair of triples of spans  $(R, S, T), (\bar{R}, \bar{S}, \bar{T})$ , a natural isomorphism:

$$\begin{aligned} \alpha_{\mu RST, \bar{R}\bar{S}\bar{T}}: (\alpha_{\mu \bar{R}, \bar{S}, \bar{T}})^* \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'}) (\otimes (\otimes \times 1)) &\Rightarrow \\ (\alpha_{\mu R, S, T})^* \text{Span}(\mathcal{B})(\alpha_{ABC}, 1) (\otimes (1 \times \otimes)), & \end{aligned}$$

consisting of, for each triple of maps of spans:



an isomorphism of maps of spans:

$$\alpha_{\mu_{f_R, f_S, f_T}} : (\varpi_{(RS)T} \cdot \pi_{(R \times S) \times T}^{(A' \times B') \times C'}, \alpha_{\mu_{\bar{R}, \bar{S}, \bar{T}}}(((f_R \times f_S) \times f_T) * 1), a \cdot \kappa^{-1} \cdot (((f_R \times f_S) \times f_T) * 1)) \\ \Rightarrow (1, (1 * (f_R \times (f_S \times f_T)))\alpha_{\mu_{R, S, T}}, (\varrho_{R(ST)} \cdot \pi_{R \times (S \times T)}^{(A \times B) \times C} \alpha_{\mu_{R, S, T}})(a \cdot \kappa^{-1})),$$

consisting of the unique 2-cell:

$$\alpha_{\mu_{f_R, f_S, f_T}} : \alpha_{\mu_{\bar{R}, \bar{S}, \bar{T}}}(((f_R \times f_S) \times f_T) * 1) \Rightarrow (1 * (f_R \times (f_S \times f_T)))\alpha_{\mu_{R, S, T}}$$

in  $\mathcal{B}$  such that:

$$\pi_{(A \times B) \times C}^{\bar{R} \times (\bar{S} \times \bar{T})} \cdot \alpha_{\mu_{f_R, f_S, f_T}} = ((\varpi_R \times \varpi_S) \times \varpi_T)^{-1} \cdot \pi_{(R \times S) \times T}^{(A' \times B') \times C'}$$

and

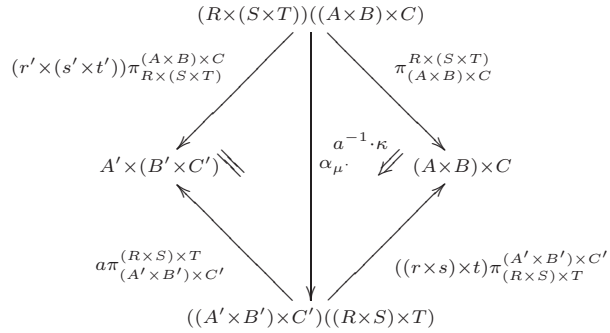
$$\pi_{R \times (S \times T)}^{(A \times B) \times C} \cdot \alpha_{\mu_{f_R, f_S, f_T}} = 1,$$

\* a strong transformation:

$$\alpha_{\mu'_{(A, B, C), (A', B', C')}} : \text{Span}(\mathcal{B})(\alpha_{ABC}, 1)(\otimes(1 \times \otimes)) \Rightarrow \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'})(\otimes(\otimes \times 1)),$$

consisting of:

· for each triple of spans  $R, S, T$ , a map of spans:



where  $\kappa := \kappa_{A \times (B \times C)}^{r \times (s \times t), 1}$  and  $\alpha_{\mu'} := \alpha_{\mu'_{RST}}$  is the unique 1-cell satisfying:

$$\pi_{(R \times S) \times T}^{(A' \times B') \times C'} \alpha_{\mu'_{RST}} = a^{-1} \pi_{R \times (S \times T)}^{(A \times B) \times C} \pi_{(A' \times B') \times C'}^{(R \times S) \times T} \alpha_{\mu'_{RST}} = ((r' \times s') \times t') a^{-1} \pi_{R \times (S \times T)}^{(A \times B) \times C}$$

and

$$\kappa_{(A' \times B') \times C'}^{1, (r' \times s') \times t'} \cdot \alpha_{\mu'_{RST}} = 1,$$

· for each pair of triples of spans  $(R, S, T), (\bar{R}, \bar{S}, \bar{T})$ , a natural isomorphism:

$$\alpha_{\mu'_{RST, \bar{R}\bar{S}\bar{T}}} : (\alpha_{\mu'_{\bar{R}, \bar{S}, \bar{T}}})^* \text{Span}(\mathcal{B})(\alpha_{ABC}, 1)(\otimes(1 \times \otimes)) \Rightarrow \\ (\alpha_{\mu'_{R, S, T}})^* \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'})(\otimes(\otimes \times 1)),$$

consisting of, for each triple of maps of spans  $f_R, f_S, f_T$ , an isomorphism of maps of spans:

$$\alpha_{\mu' f_R, f_S, f_T} : (a^{-1} \cdot \kappa \cdot (1 * (f_R \times (f_S \times f_T))), \alpha_{\mu' \bar{R}, \bar{S}, \bar{T}}(1 * (f_R \times (f_S \times f_T))), \varrho_{R(ST)} \cdot \pi_{R \times (S \times T)}^{(A \times B) \times C}) \\ \Rightarrow (a^{-1} \cdot \kappa, (((f_R \times f_S) \times f_T) * 1) \alpha_{\mu' R, S, T}, (\varrho_{(RS)T} \cdot \pi_{(R \times S) \times T}^{(A' \times B') \times C'}) \cdot \alpha_{\mu' R, S, T}),$$

consisting of the unique 2-cell:

$$\alpha_{\mu' f_R, f_S, f_T} : \alpha_{\mu' \bar{R}, \bar{S}, \bar{T}}(1 * (f_R \times (f_S \times f_T))) \Rightarrow (((f_R \times f_S) \times f_T) * 1) \alpha_{\mu' R, S, T}$$

in  $\mathcal{B}$  such that:

$$\pi_{(\bar{R} \times \bar{S}) \times \bar{T}}^{(A' \times B') \times C'} \cdot \alpha_{\mu' f_R, f_S, f_T} = 1$$

and

$$\pi_{(A' \times B') \times C'}^{(\bar{R} \times \bar{S}) \times \bar{T}} \cdot \alpha_{\mu' f_R, f_S, f_T} = ((\varrho_R \times \varrho_S) \times \varrho_T)^{-1} \cdot a^{-1} \pi_{R \times (S \times T)}^{(A \times B) \times C},$$

\* an invertible counit modification:

$$\alpha_\epsilon : \alpha_\mu \alpha_{\mu'} \Rightarrow 1_{\otimes(1 \times \otimes)},$$

consisting of, for each triple of spans  $R, S, T$ , an isomorphism of maps of spans:

$$\alpha_{\epsilon RST} : (a^{-1} \cdot \kappa, \alpha_\mu \alpha_{\mu'}, a \cdot \kappa^{-1} \cdot \alpha_{\mu'}) \Rightarrow (1, 1_{\otimes(1 \times \otimes)}, 1)$$

defined by the unique 2-cell  $\alpha_{\epsilon RST}$  in  $\mathcal{B}$  such that:

$$\pi_{(A \times B) \times C}^{R \times (S \times T)} \cdot \alpha_{\epsilon RST} = a^{-1} \cdot \kappa^{-1} r \times (s \times t), 1 \quad \text{and} \quad \pi_{R \times (S \times T)}^{(A \times B) \times C} \cdot \alpha_{\epsilon RST} = 1,$$

\* an invertible unit modification:

$$\alpha_\eta : 1_{\otimes(\otimes \times 1)} \Rightarrow \alpha_\mu \alpha_\mu$$

consisting of, for each triple of spans  $R, S, T$ , an isomorphism of maps of spans:

$$\alpha_{\eta RST} : (1, 1_{\otimes(\otimes \times 1)}, 1) \Rightarrow (a^{-1} \cdot \kappa \cdot \alpha_\mu, \alpha_\mu \alpha_\mu, a \cdot \kappa^{-1})$$

defined by the unique 2-cell  $\alpha_{\eta RST}$  in  $\mathcal{B}$  such that:

$$\pi_{(R \times S) \times T}^{(A' \times B') \times C'} \cdot \alpha_{\eta RST} = 1 \quad \text{and} \quad \pi_{(A' \times B') \times C'}^{(R \times S) \times T} \cdot \alpha_{\eta RST} = \kappa^{-1} 1, (r' \times s') \times t',$$

– an identity modification  $\alpha_\Pi$  with component equations of maps of spans:

$$(1, (1 * \chi)(\alpha_{\mu RST} * 1)(1 * \alpha_{\mu R' S' T'}), a \cdot \kappa^{-1} \cdot \pi) = (1, \alpha_{\mu(R'R)(S'S)(T'T)}(\chi * 1), a \cdot \kappa^{-1} \cdot (1 * \chi)),$$

– an identity modification  $\alpha_M$  with component equations of maps of spans:

$$(1, \alpha_\mu \iota \mathbf{r}^{-1}, a \cdot \kappa^{-1} \cdot \iota \mathbf{r}^{-1}) = (1, \iota \mathbf{l}^{-1}, 1),$$

• a tritransformation:

$$\alpha' : \otimes(1 \times \otimes) \Rightarrow \otimes(\otimes \times 1),$$

consisting of:

– for each triple  $A, B, C$  of objects in  $\text{Span}(\mathcal{B})$ , a span  $\alpha'_{ABC}$ :

$$\begin{array}{ccc} & (A \times B) \times C & \\ \swarrow 1 & & \searrow a \\ (A \times B) \times C & & A \times (B \times C) \end{array}$$



– for each two triples  $(A, B, C), (A', B', C')$  of objects in  $\text{Span}(\mathcal{B})$ , an adjoint equivalence:

$$(\alpha_\mu, \alpha_\mu^*, \alpha_\epsilon, \alpha_\eta) : \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'}) (\otimes(1 \times \otimes)) \Rightarrow \text{Span}(\mathcal{B})(\alpha_{ABC}, 1) (\otimes(\otimes \times 1)),$$

in the strict 2-category:

$$\text{Bicat}(\text{Span}(\mathcal{B})^3((A, B, C), (A', B', C')), \text{Span}(\mathcal{B})(A \times (B \times C), (A' \times B') \times C')),$$

consisting of:

\* a strong transformation:

$$\alpha_\mu^{(A, B, C), (A', B', C')} : \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'}) (\otimes(1 \times \otimes)) \Rightarrow \text{Span}(\mathcal{B})(\alpha_{ABC}, 1) (\otimes(\otimes \times 1)),$$

consisting of:

· for each triple of spans  $R, S, T$ , a map of spans:

$$\begin{array}{ccc} & ((A' \times B') \times C') (R \times (S \times T)) & \\ \swarrow \pi_{(A' \times B') \times C'}^{R \times (S \times T)} & \downarrow & \searrow (r \times (s \times t)) \pi_{R \times (S \times T)}^{(A' \times B') \times C'} \\ (A' \times B') \times C' & \xrightarrow{a^{-1} \cdot \kappa} & A \times (B \times C) \\ \nwarrow ((r' \times s') \times t') \pi_{(R \times S) \times T}^{(A \times B) \times C} & \downarrow \alpha_\mu & \nearrow a \pi_{(A \times B) \times C}^{(R \times S) \times T} \\ & ((R \times S) \times T) ((A \times B) \times C) & \end{array}$$

where  $\kappa := \kappa_{A' \times (B' \times C')}^{1, r' \times (s' \times t')}$  and  $\alpha_\mu := \alpha_{\mu RST}$  is the unique 1-cell satisfying:

$$\pi_{(A \times B) \times C}^{(R \times S) \times T} \alpha_{\mu RST} = ((r \times s) \times t) a^{-1} \pi_{R \times (S \times T)}^{(A' \times B') \times C'} \quad \pi_{(R \times S) \times T}^{(A \times B) \times C} \alpha_{\mu RST} = a^{-1} \pi_{R \times (S \times T)}^{(A' \times B') \times C'}$$

and

$$\kappa_{(A \times B) \times C}^{(r \times s) \times t, 1} \cdot \alpha_{\mu RST} = 1,$$

· for each pair of triples of spans  $(R, S, T), (\bar{R}, \bar{S}, \bar{T})$ , a natural isomorphism:

$$\begin{aligned} \alpha_{\mu RST, \bar{R}\bar{S}\bar{T}} : (\alpha_{\mu \bar{R}, \bar{S}, \bar{T}})^* \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'}) (\otimes(1 \times \otimes)) &\Rightarrow \\ (\alpha_{\mu R, S, T})^* \text{Span}(\mathcal{B})(\alpha_{ABC}, 1) (\otimes(\otimes \times 1)) & \end{aligned}$$

consisting of, for each triple of maps of spans  $f_R, f_S, f_T$ , an isomorphism of maps of spans:

$$\begin{aligned} \alpha_{\mu f_R, f_S, f_T} : (\varpi_{R(ST)} \cdot \pi_{R \times (S \times T)}^{(A' \times B') \times C'}) \cdot \alpha_{\mu \bar{R}, \bar{S}, \bar{T}}((f_R \times (f_S \times f_T)) * 1), \quad a^{-1} \cdot \kappa \cdot ((f_R \times (f_S \times f_T)) * 1)) \\ \Rightarrow (1, (1 * ((f_R \times f_S) \times f_T)) \alpha_{\mu R, S, T}, ((\varrho_{(RS)T} \cdot \pi_{(R \times S) \times T}^{(A' \times B') \times C'}) \cdot \alpha_{\mu R, S, T})(a^{-1} \cdot \kappa)), \end{aligned}$$

consisting of the unique 2-cell:

$$\alpha_{\mu f_R, f_S, f_T} : \alpha_{\mu \bar{R}, \bar{S}, \bar{T}}((f_R \times (f_S \times f_T)) * 1) \Rightarrow (1 * ((f_R \times f_S) \times f_T)) \alpha_{\mu R, S, T}$$

in  $\mathcal{B}$  such that:

$$\pi_{(\bar{R} \times \bar{S}) \times \bar{T}}^{(A \times B) \times C} \cdot \alpha_{\mu f_R, f_S, f_T} = 1$$

and

$$\pi_{(A \times B) \times C}^{(\bar{R} \times \bar{S}) \times \bar{T}} \cdot \alpha_{\mu f_R, f_S, f_T} = ((\varrho_R \times \varrho_S) \times \varrho_T)^{-1} \cdot a^{-1} \pi_{R \times (S \times T)}^{(A' \times B') \times C'},$$

\* a strong transformation:

$$\alpha_\mu : \text{Span}(\mathcal{B})(\alpha_{ABC}, 1) (\otimes(\otimes \times 1)) \Rightarrow \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'}) (\otimes(1 \times \otimes)),$$

consisting of,

· for each triple of spans  $R, S, T$ , a map of spans:

$$\begin{array}{ccc}
 & ((R \times S) \times T) \times ((A \times B) \times C) & \\
 \swarrow^{((r' \times s') \times t') \pi_{(R \times S) \times T}^{(A \times B) \times C}} & \downarrow & \searrow^{a \pi_{(A \times B) \times C}^{(R \times S) \times T}} \\
 (A' \times B') \times C' & \xrightarrow{\alpha_{\mu'}} & A \times (B \times C) \\
 \nwarrow^{\pi_{(A' \times B') \times C'}^{R \times (S \times T)}} & \downarrow & \nearrow_{(r \times (s \times t)) \pi_{R \times (S \times T)}^{(A' \times B') \times C'}} \\
 & ((A' \times B') \times C') \times (R \times (S \times T)) &
 \end{array}$$

where  $\kappa := \kappa_{(A \times B) \times C}^{(r \times s) \times t, 1}$  and  $\alpha_{\mu'} := \alpha_{\mu'}_{RST}$  is the unique 1-cell satisfying:

$$\pi_{R \times (S \times T)}^{(A' \times B') \times C'} \alpha_{\mu'}_{RST} = a \pi_{(R \times S) \times T}^{(A \times B) \times C} \pi_{(A' \times B') \times C'}^{R \times (S \times T)} \alpha_{\mu'}_{RST} = ((r' \times s') \times t') \pi_{(R \times S) \times T}^{(A \times B) \times C},$$

and

$$\kappa_{A' \times (B' \times C')}^{1, r' \times (s' \times t')} \cdot \alpha_{\mu'}_{RST} = 1,$$

· for each pair of triples of spans  $(R, S, T), (\bar{R}, \bar{S}, \bar{T})$ , a natural isomorphism:

$$\begin{aligned}
 \alpha_{\mu'}_{RST, \bar{R}\bar{S}\bar{T}} : (\alpha_{\mu'}_{\bar{R}, \bar{S}, \bar{T}})^* \text{Span}(\mathcal{B})(\alpha_{ABC}, 1)(\otimes(\otimes \times 1)) &\Rightarrow \\
 (\alpha_{\mu'}_{R, S, T})^* \text{Span}(\mathcal{B})(1, \alpha_{A'B'C'}) &(\otimes(1 \times \otimes)),
 \end{aligned}$$

consisting of, for each triple of maps of spans  $f_R, f_S, f_T$ , an isomorphism of maps of spans:

$$\begin{aligned}
 \alpha_{\mu'}_{f_R, f_S, f_T} : ((a \cdot \kappa \cdot (1 * ((f_R \times f_S) \times f_T))), \alpha_{\mu'}_{\bar{R}, \bar{S}, \bar{T}}(1 * ((f_R \times f_S) \times f_T)), \varrho_{(RS)T} \cdot \pi_{(R \times S) \times T}^{(A \times B) \times C}) \\
 \Rightarrow ((\varpi_{R(ST)} \cdot \pi_{R \times (S \times T)}^{(A \times B) \times C} \alpha_{\mu'}_{R, S, T})(a \cdot \kappa), ((f_R \times (f_S \times f_T)) * 1) \alpha_{\mu'}_{R, S, T}, 1),
 \end{aligned}$$

consisting of the unique 2-cell:

$$\alpha_{\mu'}_{f_R, f_S, f_T} : \alpha_{\mu'}_{\bar{R}, \bar{S}, \bar{T}}(1 * ((f_R \times f_S) \times f_T)) \Rightarrow ((f_R \times (f_S \times f_T)) * 1) \alpha_{\mu'}_{R, S, T}$$

in  $\mathcal{B}$  such that:

$$\pi_{\bar{R} \times (\bar{S} \times \bar{T})}^{A' \times (B' \times C')} \cdot \alpha_{\mu'}_{f_R, f_S, f_T} = 1$$

and

$$\pi_{A' \times (B' \times C')}^{\bar{R} \times (\bar{S} \times \bar{T})} \cdot \alpha_{\mu'}_{f_R, f_S, f_T} = ((\varrho_R \times \varrho_S) \times \varrho_T)^{-1} \cdot \pi_{(R \times S) \times T}^{(A \times B) \times C},$$

\* an invertible counit modification:

$$\alpha_{\epsilon_{\mu}} : \alpha_{\mu} \alpha_{\mu'} \Rightarrow 1_{\otimes(\otimes \times 1)}$$

consisting of, for each triple of spans  $R, S, T$ , an isomorphism of maps of spans:

$$\alpha_{\epsilon_{\mu}}_{RST} : (a^{-1} \cdot \kappa, \alpha_{\mu} \alpha_{\mu'}, a \cdot \kappa^{-1} \cdot \alpha_{\mu'}) \Rightarrow (1, 1_{\otimes(\otimes \times 1)}, 1)$$

defined by the unique 2-cell  $\alpha_{\epsilon_{\mu}}_{RST}$  in  $\mathcal{B}$  such that:

$$\pi_{(A \times B) \times C}^{(R \times S) \times T} \cdot \alpha_{\epsilon_{\mu}}_{RST} = a^{-1} \cdot \kappa^{-1} \pi_{A \times (B \times C)}^{r \times (s \times t), 1} \quad \text{and} \quad \pi_{(R \times S) \times T}^{(A \times B) \times C} \cdot \alpha_{\epsilon_{\mu}}_{RST} = 1,$$

\* an invertible unit modification:

$$\alpha_{\eta_{\mu}} : 1_{\otimes(1 \times \otimes)} \Rightarrow \alpha_{\mu'} \cdot \alpha_{\mu}$$

consisting of, for each triple of spans  $R, S, T$ , an isomorphism of maps of spans:

$$\alpha_{\eta_{\mu}}_{RST} : (1, 1_{\otimes(1 \times \otimes)}, 1) \Rightarrow (a^{-1} \cdot \kappa \cdot \alpha_{\mu}, \alpha_{\mu'} \cdot \alpha_{\mu}, a \cdot \kappa^{-1})$$

defined by the unique 2-cell  $\alpha_{\eta_{\mu}}_{RST}$  in  $\mathcal{B}$  such that:

$$\pi_{(R \times S) \times T}^{(A' \times B') \times C'} \cdot \alpha_{\eta_{\mu}}_{RST} = 1 \quad \text{and} \quad \pi_{(A' \times B') \times C'}^{(R \times S) \times T} \cdot \alpha_{\eta_{\mu}}_{RST} = \kappa_{(A' \times B') \times C'}^{1, (r \times s') \times t'},$$

- an identity modification  $\alpha_{\Pi}$  with component equations of maps of spans:

$$(1, (1 * \chi)(\alpha_{\mu RST} * 1)(1 * \alpha_{\mu R'S'T'}), a^{-1} \cdot \kappa \cdot \pi) = (1, \alpha_{\mu(R'R)(S'S)(T'T)}(\chi * 1), a^{-1} \cdot \kappa \cdot (1 * \chi)),$$

- an identity modification  $\alpha_M$  with component equations of maps of spans:

$$(1, \alpha_{\mu} \iota \mathbf{r}^{-1}, a^{-1} \cdot \kappa \cdot \iota \mathbf{r}^{-1}) = (1, \iota \mathbf{l}^{-1}, 1),$$

- a strict adjoint equivalence  $(\epsilon_{\alpha}, \epsilon_{\alpha}, \epsilon_{\epsilon_{\alpha}}, \eta_{\epsilon_{\alpha}})$ , in the strict 2-category:

$$\text{Bicat}(\text{Span}(\mathcal{B})^3((A, B, C), (A', B', C')), \text{Span}(\mathcal{B})(A \times (B \times C), A' \times (B' \times C'))),$$

consisting of:

- a trimodification:

$$\epsilon_{\alpha} : \alpha \alpha' \Rightarrow I_{\otimes(1 \times \otimes)}$$

consisting of:

- \* for each triple of objects  $A, B, C \in \text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc} & ((A \times B) \times C)((A \times B) \times C) & \\ \swarrow a\pi_{(A \times B) \times C}^{(A \times B) \times C} & \downarrow & \searrow a\pi_{(A \times B) \times C}^{(A \times B) \times C} \\ A \times (B \times C) & \xrightarrow{\epsilon_{\alpha}} & A \times (B \times C) \\ \nwarrow 1 & & \nearrow 1 \\ & A \times (B \times C) & \end{array}$$

where  $\epsilon_{\alpha} := \epsilon_{\alpha A, B, C} = a\pi_{(A \times B) \times C}^{(A \times B) \times C}$ ,

- \* for each pair of triples of objects  $(A, B, C), (A', B', C')$ , an identity modification:

$$m_{\epsilon_{\alpha}} : (\epsilon_{\alpha(A, B, C)})^* \alpha \alpha' \Rightarrow I_{\otimes(1 \times \otimes)}(\epsilon_{\alpha(A', B', C')})^*$$

consisting of, for each triple of spans  $R, S, T$ , an equation of maps of spans:

$$\begin{aligned} & (1, (\epsilon_{\alpha ABC} * 1_{R(ST)})(1_{(AB)C} * \alpha_{\mu RST})(\alpha_{\mu RST} * 1_{(A'B')C'}), a \cdot \kappa^{-1} \cdot \pi(\alpha_{\mu RST} * 1_{(A'B')C'})) \\ & = (1, I_{R(ST)}(1_{R(ST)} * \epsilon_{\alpha A'B'C'}), 1), \end{aligned}$$

- a trimodification:

$$\epsilon'_{\alpha} : I_{\otimes(1 \times \otimes)} \Rightarrow \alpha \alpha'$$

consisting of:

- \* for each triple of objects  $A, B, C \in \text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc} & A \times (B \times C) & \\ \swarrow 1 & \downarrow & \searrow 1 \\ A \times (B \times C) & \xrightarrow{\epsilon'_{\alpha}} & A \times (B \times C) \\ \nwarrow a\pi_{(A \times B) \times C}^{(A \times B) \times C} & & \nearrow a\pi_{(A \times B) \times C}^{(A \times B) \times C} \\ & ((A \times B) \times C)((A \times B) \times C) & \end{array}$$

where  $\epsilon'_{\alpha} := \epsilon'_{\alpha A, B, C}$  is the unique 1-cell in  $\mathcal{B}$  satisfying:

$$\pi_{(A \times B) \times C}^{(A \times B) \times C} \epsilon'_{\alpha A, B, C} = a^{-1},$$

\* for each pair of triples of objects  $(A, B, C), (A', B', C')$ , an identity modification:

$$m_{\epsilon_\alpha} : (m_{\epsilon_\alpha(A, B, C)})^* I_{\otimes(1 \times \otimes)} \Rightarrow \alpha \alpha^* (m_{\epsilon_\alpha(A', B', C')})^*$$

consisting of, for each triple of spans  $R, S, T$ , an equation of maps of spans:

$$(1, (\epsilon_{\alpha ABC} * 1_{R(ST)}) I_{R(ST)}, 1) =$$

$$(1, (1_{(AB)C} * \alpha_{\mu_{RST}})(\alpha_{\mu_{RST}}^* 1_{(A'B')C'})(1_{R(ST)} * \epsilon_{\alpha A'B'C'}), a \cdot \kappa^{-1} \cdot \pi(\alpha_{\mu_{RST}}^* 1_{(A'B')C'})),$$

– an identity perturbation:

$$\epsilon_{\epsilon_\alpha} : \epsilon_\alpha \epsilon_\alpha^* \Rightarrow 1_{I_{\otimes(1 \times \otimes)}}$$

consisting of, for each triple of objects  $A, B, C$ , an equation of maps of spans:

$$\epsilon_{\epsilon_\alpha ABC} : \epsilon_{\alpha ABC} \epsilon_{\alpha ABC}^* \Rightarrow 1_{I_{\otimes(1 \times \otimes)} ABC},$$

– an identity perturbation:

$$\eta_{\epsilon_\alpha} : 1_{\alpha \alpha^*} \Rightarrow \epsilon_\alpha^* \epsilon_\alpha$$

consisting of, for each triple of objects  $A, B, C$ , an equation of maps of spans:

$$\eta_{\epsilon_\alpha ABC} : 1_{\alpha ABC} \alpha_{ABC}^* \Rightarrow \epsilon_{\alpha ABC}^* \epsilon_{\alpha ABC},$$

- a strict adjoint equivalence  $(\eta_\alpha, \eta_\alpha^*, \epsilon_{\eta_\alpha}, \eta_{\eta_\alpha})$ , in the strict 2-category:

$$\text{Bicat}(\text{Span}(\mathcal{B})^3((A, B, C), (A', B', C')), \text{Span}(\mathcal{B})((A \times B) \times C, (A' \times B') \times C')),$$

consisting of:

– a trimodification:

$$\eta_\alpha : \alpha^* \alpha \Rightarrow I_{\otimes(\otimes \times 1)}$$

consisting of:

\* for each triple of objects  $A, B, C \in \text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc} & ((A \times B) \times C)((A \times B) \times C) & \\ \pi_{(A \times B) \times C}^{(A \times B) \times C} \swarrow & \downarrow & \searrow \pi_{(A \times B) \times C}^{(A \times B) \times C} \\ (A \times B) \times C & \eta_\alpha & (A \times B) \times C \\ \uparrow 1 & & \downarrow 1 \\ & (A \times B) \times C & \end{array}$$

where  $\eta_\alpha := \eta_{\alpha A, B, C} = \pi_{(A \times B) \times C}^{(A \times B) \times C}$

\* for each pair of triples of objects  $(A, B, C), (A', B', C')$ , an identity modification:

$$m_{\eta_\alpha} : (\eta_{\alpha(A, B, C)})^* \alpha^* \alpha \Rightarrow I_{\otimes(\otimes \times 1)}(\eta_{\alpha A', B', C'})^*$$

consisting of, for each triple of spans  $R, S, T$ , an equation of maps of spans:

$$\begin{aligned} & (1, (\eta_{\alpha ABC} * 1_{(RS)T})(1_{(AB)C} * \alpha_{\mu_{RST}})(\alpha_{\mu_{RST}}^* 1_{(A'B')C'}), a^{-1} \cdot \kappa \cdot \pi(\alpha_{\mu_{RST}}^* 1_{(A'B')C'})) \\ & = (1, I_{(RS)T}(1_{(RS)T} * \eta_{\alpha A'B'C'}), 1), \end{aligned}$$

– a trimodification:

$$\eta_\alpha^* : I_{\otimes(\otimes \times 1)} \Rightarrow \alpha^* \alpha$$

consisting of:

\* for each triple of objects  $A, B, C \in \text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccccc}
 & (A \times B) \times C & & & \\
 & \downarrow & & \downarrow & \\
 (A \times B) \times C & \xleftarrow{1} & & \xrightarrow{1} & (A \times B) \times C \\
 & \searrow & \eta_\alpha & \swarrow & \\
 & \pi_{(A \times B) \times C}^{(A \times B) \times C} & & \pi_{(A \times B) \times C}^{(A \times B) \times C} & \\
 & ((A \times B) \times C)((A \times B) \times C) & & & 
 \end{array}$$

where  $\eta_\alpha := \eta_{\alpha A, B, C}$  is the unique 1-cell in  $\mathcal{B}$  satisfying:

$$\pi_{(A \times B) \times C}^{(A \times B) \times C} \eta_{\alpha A, B, C} = 1,$$

\* for each pair of triples of objects  $(A, B, C), (A', B', C')$ , an identity modification:

$$m_{\eta_\alpha} : (m_{\eta_\alpha(A, B, C)})^* I_{\otimes(\otimes \times 1)} \Rightarrow \alpha' \alpha (m_{\eta_\alpha(A', B', C')})^*$$

consisting of, for each triple of spans  $R, S, T$ , an equation of maps of spans:

$$(1, (\eta_{\alpha ABC} * 1_{(RS)T}) I_{(RS)T}, 1) =$$

$$(1, (1_{(AB)C} * \alpha'_\mu RST)(\alpha_\mu RST * 1_{(A'B')C'}) (1_{(RS)T} * \eta_{\alpha A' B' C'}), a^{-1} \cdot \kappa \cdot \pi(\alpha_\mu RST * 1_{(A'B')C'})),$$

– an identity perturbation:

$$\epsilon_{\eta_\alpha} : \eta_\alpha \eta_\alpha \Rightarrow 1_{I_{\otimes(\otimes \times 1)}}$$

consisting of, for each triple of objects  $A, B, C$ , an equation of maps of spans:

$$\epsilon_{\eta_\alpha ABC} : \eta_{\alpha ABC} \eta_{\alpha ABC} \Rightarrow 1_{I_{\otimes(1 \times \otimes) ABC}},$$

– an identity perturbation:

$$\eta_{\eta_\alpha} : 1_{I_{\otimes(\otimes \times 1)}} \Rightarrow \eta_\alpha \eta_\alpha$$

consisting of, for each triple of objects  $A, B, C$ , an equation of maps of spans:

$$\eta_{\eta_\alpha ABC} : 1_{\alpha_{ABC} \alpha_{ABC}} \Rightarrow \eta_{\alpha ABC} \eta_{\alpha ABC},$$

• an identity perturbation:

$$\Phi_\alpha : (1, 1(1 * \epsilon_\alpha) \mathbf{a}(\eta_\alpha * 1) \mathbf{r}^{-1}, \kappa^{-1} \cdot (1 * \epsilon_\alpha) \mathbf{a}(\eta_\alpha * 1) \mathbf{r}^{-1}) \Rightarrow (1, 1, 1)$$

consisting of, for each triple of objects  $A, B, C \in \text{Span}(\mathcal{B})$ , an equation of maps of spans,

• and, an identity perturbation:

$$\Psi_\alpha : (\kappa \cdot (\epsilon_\alpha * 1) \mathbf{a}^{-1}(1 * \eta_\alpha) \mathbf{l}^{-1}, \mathbf{r}(\epsilon_\alpha * 1) \mathbf{a}^{-1}(1 * \eta_\alpha) \mathbf{l}^{-1}, 1) \Rightarrow (1, 1, 1)$$

consisting of, for each triple of objects  $A, B, C \in \text{Span}(\mathcal{B})$ , an equation of maps of spans.

*Proof.* We check that  $\alpha$  and  $\alpha'$  are tritransformations, that  $\epsilon_\alpha$  and  $\eta_\alpha$  are adjoint equivalences, and that  $\Phi_\alpha$  and  $\Psi_\alpha$  are invertible perturbations. Finally, we verify the axioms of a biadjoint biequivalence.

We first check that  $\alpha$  is a tritransformation and the proof for  $\alpha'$  follows similarly. We need to verify that  $(\alpha_\mu, \alpha_\mu, \alpha_\epsilon, \alpha_\eta)$  is an adjoint equivalence of strong transformations and modifications.

Naturality for  $\alpha_{\mu(R, S, T), (\bar{R}, \bar{S}, \bar{T})}$  is a straightforward calculation to showing that, for each triple of maps of maps of spans  $(\sigma_R, \sigma_S, \sigma_T)$ , the equation:

$$\alpha_{\mu_{g_R, g_S, g_T}} 1_{\alpha_\mu} (\sigma_{(RS)T} * 1) = (1 * \sigma_{R(ST)}) 1_{\alpha_\mu} \alpha_{\mu_{f_R, f_S, f_T}}$$

holds.

Since the monoidal product is locally strict, the transformation axioms simplify. One is a simple calculation and the other is immediate. We have shown that  $\alpha_\mu$  is a strong transformation. Similarly,  $\alpha_{\mu^*}$  is a strong transformation.

The 2-cell equation:

$$((r \times (s \times t)) \cdot (\pi \cdot \alpha_{\epsilon RST}))(\kappa_{A \times (B \times C)}^{r \times (s \times t), 1} \cdot \alpha_\mu \alpha_{\mu^*}) = (\kappa_{A \times (B \times C)}^{r \times (s \times t), 1} \cdot 1)(1 \cdot (\pi \cdot \alpha_{\epsilon RST}))$$

allows us to apply the universal property in defining the component isomorphisms of spans of the counit. A similar equation gives the unit isomorphism. The modification axiom for  $\alpha_\epsilon$  is the following equation of 2-cells:

$$1_{f_R \times (f_S \times f_T)}(\alpha_{\epsilon \bar{R}, \bar{S}, \bar{T}} 1_{1*(f_R \times (f_S \times f_T))}) = (1_{1*(f_R \times (f_S \times f_T))} \alpha_{\epsilon R, S, T})(\alpha_{\mu f_R, f_S, f_T} 1_{\alpha_{\mu^* R, S, T}})(1_{\alpha_{\mu R, S, T}} \alpha_{\mu^* f_R, f_S, f_T})$$

which is easily verified by definitions. A similar modification axiom can be checked for  $\alpha_\eta$ .

The transformations and modifications form an adjoint equivalence in a strict 2-category and the axioms reduce to the equations:

$$(1_{\alpha_{\mu^*}} \alpha_\epsilon)(\alpha_\eta 1_{\alpha_{\mu^*}}) = 1_{\alpha_{\mu^*}}$$

and

$$(\alpha_\eta 1_{\alpha_\mu})(1_{\alpha_\mu} \alpha_\epsilon) = 1_{\alpha_\mu},$$

which are verified by simple calculations.

The modification axioms are immediate for the collections of identity cells  $\alpha_\Pi$  and  $\alpha_M$ .

The tritransformation axioms are immediate since all modifications cells from  $\text{Span}(\mathcal{B})$ , the monoidal product, and  $\alpha$  are identities. It follows that  $\alpha$  and similarly  $\alpha^*$  are tritransformations.

Next we check that  $\epsilon_\alpha$  and  $\epsilon'_\alpha$  are trimodifications and the 1-cells of an adjoint equivalence. Similar results will hold for  $\eta_\alpha$  and  $\eta'_\alpha$ .

The modification axiom is immediate since the 2-cells  $m_{\epsilon_\alpha RST}$  are identities. The trimodification axioms are also immediately satisfied since these 2-cells and the modification components  $\alpha_\Pi$ ,  $\alpha_M$ , and the analogous modifications for  $\alpha^*$  and  $I_{\otimes(1 \times \otimes)}$  are all identities. It follows that  $\epsilon_\alpha$ , and similarly  $\epsilon'_\alpha$  are trimodifications.

The counit and unit perturbations  $\epsilon_{\epsilon_\alpha}$  and  $\eta_{\epsilon_\alpha}$  are each identities and thus trivially satisfy the perturbation axioms. Further the adjoint equivalence axioms are immediately satisfied. It follows that  $(\epsilon_\alpha, \epsilon'_\alpha, \epsilon_{\epsilon_\alpha}, \eta_{\epsilon_\alpha})$ , and similarly  $(\eta_\alpha, \eta'_\alpha, \epsilon_{\eta_\alpha}, \eta_{\eta_\alpha})$ , are adjoint equivalences.

Finally, the axioms of a biadjoint biequivalence will be satisfied since the perturbations  $\Phi_\alpha$  and  $\Psi_\alpha$  are identities and the modifications of  $\text{Span}(\mathcal{B})$  are all identities. To check these equations precisely we need to specify the tricategory structure on  $\text{Tricat}(\text{Span}^3(\mathcal{B}), \text{Span}(\mathcal{B}))$ , but we do not include these details here.  $\square$

## Monoidal Left Unitor

The monoidal left unitor is a biadjoint biequivalence consisting of tritransformations, adjoint equivalences of trimodifications and perturbations, and coherence perturbations consisting of isomorphisms of maps of spans.

The left unitor for the product of an object  $A \in \mathcal{B}$  and the unit  $1 \in \mathcal{B}$  is the 1-cell:

$$\lambda_A: 1 \times A \rightarrow A$$

in  $\mathcal{B}$  defined as the projection:

$$\tilde{\pi}_A^1: 1 \times A \rightarrow A$$

with 1-cell:

$$\tilde{\pi}_{1 \times A}^1: A \rightarrow 1 \times A$$

defined as the unique inverse 1-cell in  $\mathcal{B}$ .

**Proposition 45.** *There is a biadjoint biequivalence:*

$$(\lambda, \lambda', \epsilon_\lambda, \eta_\lambda, \Phi_\lambda, \Psi_\lambda): \otimes(I \times 1) \Rightarrow \otimes 1,$$

in a ‘tricategory’  $\text{Tricat}(\text{Span}(\mathcal{B}), \text{Span}(\mathcal{B}))$ , consisting of:

- a tritransformation:

$$\lambda: \otimes(I \times 1) \Rightarrow \otimes 1,$$

consisting of:

– for each object  $A \in \text{Span}(\mathcal{B})$ , a span  $\lambda_A$ :

$$\begin{array}{ccc} & 1 \times A & \\ \tilde{\pi}_A^1 \swarrow & & \searrow 1 \\ A & & 1 \times A \end{array}$$

– for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , an adjoint equivalence:

$$(\lambda_\mu, \lambda_\mu^*, \lambda_\epsilon, \lambda_\eta): \text{Span}(\mathcal{B})(1, \lambda_B)(\otimes(I \times 1)) \Rightarrow \text{Span}(\mathcal{B})(\lambda_A, 1)1,$$

in the strict 2-category:

$$\text{Bicat}(\text{Span}(\mathcal{B})(A, B), \text{Span}(\mathcal{B})(1 \times A, B))$$

consisting of:

\* a strong transformation:

$$\lambda_{\mu_{A,B}}: \text{Span}(\mathcal{B})(1, \lambda_B)(\otimes(I \times 1)) \Rightarrow \text{Span}(\mathcal{B})(\lambda_A, 1)1,$$

consisting of:

· for each span:

$$\begin{array}{ccc} & R & \\ q \swarrow & & \searrow p \\ B & & A \end{array}$$

a map of spans:

$$\begin{array}{ccccc} & (1 \times B)(1 \times R) & & & \\ \tilde{\pi}_B^1 \pi_{1 \times B}^{1 \times R} \swarrow & \downarrow & \searrow (1 \times p) \pi_{1 \times R}^{1 \times B} & & \\ B & \xrightarrow{\tilde{\pi}_B^1 \cdot \kappa^{-1}} & 1 \times A & & \\ q \pi_R^{1 \times A} \swarrow & \downarrow \lambda_\mu & \searrow \pi_{1 \times A}^R & & \\ & R(1 \times A) & & & \end{array}$$

where  $\tilde{\pi}_B^1 \cdot \kappa := \tilde{\pi}_B^1 \cdot \kappa_{1 \times B}^{1, 1 \times q}$  and  $\lambda_\mu := \lambda_{\mu_R}$  is the unique 1-cell satisfying:

$$\pi_{1 \times A}^R \lambda_{\mu_R} = (1 \times p) \pi_{1 \times R}^{1 \times B} \quad \pi_R^{1 \times A} \lambda_{\mu_R} = \tilde{\pi}_R^1 \pi_{1 \times R}^{1 \times B}$$

and

$$\kappa_A^{p, \tilde{\pi}_A^1} \cdot \lambda_{\mu_R} = 1,$$

· for each pair of spans  $R, \bar{R}$ , a natural isomorphism:

$$\lambda_{\mu_{R, \bar{R}}}: (\lambda_{\mu_{\bar{R}}})^* \text{Span}(\mathcal{B})(1, \lambda_B)(\otimes(I \times 1)) \Rightarrow (\lambda_{\mu_R})^* \text{Span}(\mathcal{B})(\lambda_A, 1)1,$$

consisting of, for each map of spans  $f_R$ , an isomorphism of maps of spans:

$$\begin{aligned} \lambda_{\mu_{f_R}}: ((1 \times \varpi_R) \cdot \pi, \lambda_{\mu_{\bar{R}}}((1 \times f_R) * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot ((1 \times f_R) * 1)) \\ \Rightarrow (1, (1 * f_R) \lambda_{\mu_R}, (\varrho_R \cdot \pi \lambda_{\mu_R})(\tilde{\pi} \cdot \kappa^{-1})), \end{aligned}$$

consisting of the unique 2-cell:

$$\lambda_{\mu_{f_R}}: \lambda_{\mu_{\bar{R}}}((1 \times f_R) * 1) \Rightarrow (1 * f_R) \lambda_{\mu_R}$$

in  $\mathcal{B}$  such that:

$$\pi_{1 \times A}^{\bar{R}} \cdot \lambda_{\mu_{f_R}} = (1 \times \varpi_R)^{-1} \cdot \pi_{1 \times R}^{1 \times B} \quad \text{and} \quad \pi_{\bar{R}}^{1 \times A} \cdot \lambda_{\mu_{f_R}} = 1,$$

\* a strong transformation:

$$\lambda_{\mu^* A, B}: \text{Span}(\mathcal{B})(\lambda_A, 1)1 \Rightarrow \text{Span}(\mathcal{B})(1, \lambda_B)(\otimes(I \times 1)),$$

consisting of:

· for each span  $R$ , a map of spans:

$$\begin{array}{ccc} & R(1 \times A) & \\ q\pi_R^{1 \times A} \swarrow & \downarrow \lambda_{\mu^*} & \searrow \pi_{1 \times A}^R \\ B & & 1 \times A \\ \tilde{\pi}_B^1 \pi_{1 \times B}^{1 \times R} \swarrow & & \searrow (1 \times p)\pi_{1 \times R}^{1 \times B} \\ & (1 \times B)(1 \times R) & \end{array}$$

$\tilde{\pi} \cdot \kappa$  (2-cell from  $\lambda_{\mu^*}$  to  $\pi_{1 \times A}^R$ )

where  $\tilde{\pi} \cdot \kappa := \tilde{\pi}_{1 \times A}^A \cdot \kappa_A^{p, \tilde{\pi}_A^1}$  and  $\lambda_{\mu^*} := \lambda_{\mu^* R}$  is the unique 1-cell satisfying:

$$\pi_{1 \times R}^{1 \times B} \lambda_{\mu^* R} = \tilde{\pi}_{1 \times R}^R \pi_R^{1 \times A} \quad \pi_{1 \times B}^{1 \times R} \lambda_{\mu^* R} = (1 \times q) \tilde{\pi}_{1 \times R}^R \pi_R^{1 \times A}$$

and

$$\kappa_{1 \times B}^{1, 1 \times q} \cdot \lambda_{\mu^* R} = 1,$$

· for each pair of spans  $R, \bar{R}$ , a natural isomorphism:

$$\lambda_{\mu^* R, \bar{R}}: (\lambda_{\mu^* R})^* \text{Span}(\mathcal{B})(\lambda_A, 1)1 \Rightarrow (\lambda_{\mu^* R})^* \text{Span}(\mathcal{B})(1, \lambda_B)(\otimes(I \times 1)),$$

consisting of, for each map of spans  $f_R$ , an isomorphism of maps of spans:

$$\begin{aligned} \lambda_{\mu^* f_R}: (\tilde{\pi} \cdot \kappa \cdot (1 * f_R), \lambda_{\mu^* \bar{R}}(1 * f_R), \varrho_R \cdot \pi) \Rightarrow \\ (((1 \times \varpi_R) \cdot \pi \lambda_{\mu^* R})(\tilde{\pi} \cdot \kappa), ((1 \times f_R) * 1) \lambda_{\mu^* R}, 1), \end{aligned}$$

consisting of the unique 2-cell:

$$\lambda_{\mu^* f_R}: \lambda_{\mu^* \bar{R}}(1 * f_R) \Rightarrow ((1 \times f_R) * 1) \lambda_{\mu^* R}$$

in  $\mathcal{B}$  such that:

$$\pi_{1 \times \bar{R}}^{1 \times B} \cdot \lambda_{\mu^* f_R} = 1 \quad \text{and} \quad \pi_{1 \times B}^{1 \times \bar{R}} \cdot \lambda_{\mu^* f_R} = \tilde{\pi}_{1 \times B}^B \cdot \varrho_R^{-1} \cdot \pi_R^{1 \times A},$$

\* an invertible counit modification:

$$\lambda_\epsilon: \lambda_\mu \lambda_{\mu^*} \Rightarrow 1$$

consisting of, for each span  $R$ , an isomorphism of maps of spans:

$$\lambda_{\epsilon R}: (\tilde{\pi} \cdot \kappa, \lambda_\mu \lambda_{\mu^*}, \tilde{\pi} \cdot \kappa^{-1} \cdot \lambda_{\mu^*}) \Rightarrow (1, 1, 1)$$

defined by the unique 2-cell  $\lambda_{\epsilon R}$  in  $\mathcal{B}$  such that:

$$\pi_{1 \times A}^R \cdot \lambda_{\epsilon R} = \tilde{\pi}_{1 \times A}^A \cdot \kappa_A^{p, \tilde{\pi}_A^1}^{-1} \quad \text{and} \quad \pi_R^{1 \times A} \cdot \lambda_{\epsilon R} = 1,$$

\* an invertible unit modification:

$$\lambda_\eta: 1 \Rightarrow \lambda_{\mu^*} \lambda_\mu$$

consisting of, for each span  $R$ , an isomorphism of maps of spans:

$$\lambda_{\eta R}: (1, 1, 1) \Rightarrow (\kappa \cdot \lambda_\mu, \lambda_{\mu^*} \lambda_\mu, \tilde{\pi} \cdot \kappa^{-1})$$

defined by the unique 2-cell  $\lambda_{\eta R}$  in  $\mathcal{B}$  such that:

$$\pi_{1 \times R}^{1 \times B} \cdot \lambda_{\eta R} = 1 \quad \text{and} \quad \pi_{1 \times B}^{1 \times R} \cdot \lambda_{\eta R} = \kappa_{1 \times B}^{1, 1 \times q}^{-1},$$



– an identity modification  $\lambda_\Pi$  with component equations of maps of spans:

$$(1, \chi(\lambda_{\mu_R} * 1)(1 * \lambda_{\mu_S}), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi) = (1, \lambda_{\mu_{SR}}(\chi * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot (\chi * 1)),$$

– an identity modification  $\lambda_M$  with component equations of maps of spans:

$$(1, \lambda_{\mu_A} \iota \mathbf{r}^{-1}, \tilde{\pi} \cdot \kappa^{-1} \cdot \iota \mathbf{r}^{-1}) = (1, \iota \mathbf{l}^{-1}, 1),$$

• a tritransformation:

$$\lambda^\cdot : 1 \Rightarrow \otimes(I \times 1),$$

consisting of:

– for each object  $A \in \text{Span}(\mathcal{B})$ , a span  $\lambda^\cdot_A$ :

$$\begin{array}{ccc} & 1 \times A & \\ \swarrow 1 & & \searrow \tilde{\pi}_A^1 \\ 1 \times A & & A \end{array}$$

– for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , an adjoint equivalence:

$$(\lambda^\cdot_\mu, \lambda^\cdot_\mu, \lambda^\cdot_\epsilon, \lambda^\cdot_\eta) : \text{Span}(\mathcal{B})(1, \lambda^\cdot_B)1 \Rightarrow \text{Span}(\mathcal{B})(\lambda^\cdot_A, 1)(\otimes(I \times 1)),$$

in the strict 2-category:

$$\text{Bicat}(\text{Span}(\mathcal{B})(A, B), \text{Span}(\mathcal{B})(A, 1 \times B)),$$

consisting of:

\* a strong transformation:

$$\lambda^\cdot_{\mu_{A,B}} : \text{Span}(\mathcal{B})(1, \lambda^\cdot_B)1 \Rightarrow \text{Span}(\mathcal{B})(\lambda^\cdot_A, 1)(\otimes(I \times 1)),$$

consisting of:

· for each span  $R$ , a map of spans:

$$\begin{array}{ccccc} & (1 \times B)R & & & \\ \pi_{1 \times B}^R \swarrow & \downarrow & \searrow p\pi_{1 \times B}^R & & \\ 1 \times B & \xrightarrow{\tilde{\pi}_{1 \times B}^B \cdot \kappa^{-1}} & \lambda^\cdot_\mu & \xrightarrow{\parallel} & A \\ (1 \times q)\pi_{1 \times R}^{1 \times A} \swarrow & \downarrow & \searrow \tilde{\pi}_A^1 \pi_{1 \times A}^{1 \times R} & & \\ & (1 \times R)(1 \times A) & & & \end{array}$$

where  $\tilde{\pi}_{1 \times B}^B \cdot \kappa := \tilde{\pi}_{1 \times B}^B \cdot \kappa_B^{1,q}$  and  $\lambda^\cdot_\mu := \lambda^\cdot_{\mu_R}$  is the unique 1-cell satisfying:

$$\pi_{1 \times A}^{1 \times R} \lambda^\cdot_{\mu_R} = (1 \times p) \tilde{\pi}_{1 \times R}^R \pi_{1 \times B}^{1 \times B} \quad \pi_{1 \times R}^{1 \times A} \lambda^\cdot_{\mu_R} = \tilde{\pi}_{1 \times R}^R \pi_{1 \times B}^{1 \times B}$$

and

$$\kappa_{1 \times A}^{1 \times p, 1} \cdot \lambda^\cdot_{\mu_R} = 1,$$

· for each pair of spans  $R, \bar{R}$ , a natural isomorphism:

$$\lambda^\cdot_{\mu_R, \bar{R}} : (\lambda^\cdot_{\mu_{\bar{R}}})_* \text{Span}(\mathcal{B})(1, \lambda^\cdot_B)1 \Rightarrow (\lambda^\cdot_{\mu_R})^* \text{Span}(\mathcal{B})(\lambda^\cdot_A, 1)(\otimes(I \times 1)),$$

consisting of, for each map of spans  $f_R$ , an isomorphism of maps of spans:

$$\lambda^\cdot_{\mu_{f_R}} : (\varpi_R \cdot \pi, \lambda^\cdot_{\mu_{\bar{R}}}(f_R * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot (f_R * 1)) \Rightarrow$$

$$(1, (1 * (1 \times f_R))\lambda_{\mu_R}, ((1 \times \varrho_R) \cdot \pi\lambda_{\mu_R})(\tilde{\pi} \cdot \kappa^{-1})),$$

consisting of the unique 2-cell:

$$\lambda_{\mu_{f_R}} : \lambda_{\mu_{\bar{R}}}(f_R * 1) \Rightarrow (1 * (1 \times f_R))\lambda_{\mu_R}$$

in  $\mathcal{B}$  such that:

$$\pi_{1 \times A}^{1 \times \bar{R}} \cdot \lambda_{\mu_{f_R}} = \tilde{\pi}_{1 \times A}^A \cdot \varpi_R \cdot \pi_R^{1 \times B} \quad \text{and} \quad \pi_{1 \times \bar{R}}^{1 \times A} \cdot \lambda_{\mu_{f_R}} = 1,$$

\* a strong transformation:

$$\lambda_{\mu_{A,B}} : \text{Span}(\mathcal{B})(\lambda_A, 1)(\otimes(I \times 1)) \Rightarrow \text{Span}(\mathcal{B})(1, \lambda_B)1,$$

consisting of:

· for each span  $R$ , a map of spans:

$$\begin{array}{ccccc} & & (1 \times R)(1 \times A) & & \\ & \swarrow (1 \times q)\pi_{1 \times R}^{1 \times A} & \downarrow & \searrow \tilde{\pi}_A^1 \pi_{1 \times A}^{1 \times R} & \\ 1 \times B & \xrightarrow{\quad \quad} & & & A \\ & \nwarrow \pi_{1 \times B}^R & \downarrow & \nearrow p\pi_R^{1 \times B} & \\ & & (1 \times B)R & & \end{array}$$

$\lambda_{\mu'}^1 \cdot \kappa$

where  $\tilde{\pi}_A^1 \cdot \kappa := \tilde{\pi}_A^1 \cdot \kappa_{1 \times A}^{p,1}$  and  $\lambda_{\mu'} := \lambda_{\mu_R}$  is the unique 1-cell satisfying:

$$\pi_R^{1 \times B} \lambda_{\mu_R} = \tilde{\pi}_R^1 \pi_{1 \times R}^{1 \times A} \quad \pi_{1 \times B}^R \lambda_{\mu_R} = (1 \times q)\pi_{1 \times R}^{1 \times A}$$

and

$$\kappa_B^{\tilde{\pi}_B^1, q} \cdot \lambda_{\mu_R} = 1,$$

· for each pair of spans  $R, \bar{R}$ , a natural isomorphism:

$$\lambda_{\mu_{R, \bar{R}}} : (\lambda_{\mu_{\bar{R}}})_* \text{Span}(\lambda_A, 1)(\otimes(I \times 1)) \Rightarrow (\lambda_{\mu_R})^* \text{Span}(1, \lambda_B)1,$$

consisting of, for each map of spans  $f_R$ , an isomorphism of maps of spans:

$$\begin{aligned} \lambda_{\mu_{f_R}} : & (\tilde{\pi} \cdot \kappa \cdot (1 * (1 \times f_R)), \lambda_{\mu_{\bar{R}}}(1 * (1 \times f_R)), (1 \times \varrho_R) \cdot \pi\lambda_{\mu_{\bar{R}}}) \\ \Rightarrow & ((\varpi_R \cdot \pi\lambda_{\mu_R})(\tilde{\pi} \cdot \kappa), (f_R * 1)\lambda_{\mu_R}, 1), \end{aligned}$$

consisting of the unique 2-cell:

$$\lambda_{\mu_{f_R}} : \lambda_{\mu_{\bar{R}}}(1 * (1 \times f_R)) \Rightarrow (f_R * 1)\lambda_{\mu_R}$$

in  $\mathcal{B}$  such that:

$$\pi_{\bar{R}}^{1 \times B} \cdot \lambda_{\mu_{f_R}} = 1 \quad \text{and} \quad \pi_{1 \times B}^{\bar{R}} \cdot \lambda_{\mu_{f_R}} = (1 \times \varrho_R)^{-1} \cdot \pi_{1 \times R}^{1 \times A},$$

\* an invertible counit modification:

$$\lambda_{\epsilon} : \lambda_{\mu} \lambda_{\mu'} \Rightarrow 1_1$$

consisting of, for each span  $R$ , an isomorphism of maps of spans:

$$\lambda_{\epsilon_R} : (\tilde{\pi} \cdot \kappa, \lambda_{\mu} \lambda_{\mu'}, \tilde{\pi} \cdot \kappa^{-1} \cdot \lambda_{\mu'}) \Rightarrow (1, 1_1, 1)$$

defined by the unique 2-cell  $\lambda_{\epsilon_R}$  in  $\mathcal{B}$  such that:

$$\pi_{1 \times A}^{1 \times R} \cdot \lambda_{\epsilon_R} = \kappa_{1 \times A}^{1 \times p, 1} \quad \text{and} \quad \pi_{1 \times R}^{1 \times A} \cdot \lambda_{\epsilon_R} = 1,$$

\* an invertible unit modification:

$$\lambda_{\eta} : 1 \Rightarrow \lambda_{\mu} \cdot \lambda_{\mu}$$

consisting of, for each span  $R$ , an isomorphism of maps of spans:

$$\lambda_{\eta R} : (1, 1, 1) \Rightarrow (\tilde{\pi} \cdot \kappa \cdot \lambda_{\mu}, \lambda_{\mu} \cdot \lambda_{\mu}, \tilde{\pi} \cdot \kappa^{-1})$$

defined by the unique 2-cell  $\lambda_{\eta R}$  in  $\mathcal{B}$  such that:

$$\pi_R^{1 \times B} \cdot \lambda_{\eta R} = 1 \quad \text{and} \quad \pi_{1 \times B}^R \cdot \lambda_{\eta R} = \tilde{\pi}_{1 \times B}^B \cdot \kappa_B^{\tilde{\pi}_B^1, q^{-1}},$$

– an identity modification  $\lambda_{\Pi}$  with component equations of maps of spans:

$$(1, \chi(\lambda_{\mu R} * 1)(1 * \lambda_{\mu S}), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi) = (1, \lambda_{\mu SR}(\chi * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot (\chi * 1)),$$

– an identity modification  $\lambda_M$  with component equations of maps of spans:

$$(1, \lambda_{\mu A} \iota \mathbf{r}^{-1}, \tilde{\pi} \cdot \kappa^{-1} \cdot \iota \mathbf{r}^{-1}) = (1, \iota \mathbf{l}^{-1}, 1),$$

• a strict adjoint equivalence  $(\epsilon_{\lambda}, \epsilon_{\lambda}^{\dagger}, \epsilon_{\epsilon_{\lambda}}, \eta_{\epsilon_{\lambda}})$ : consisting of:

– a trimodification:

$$\epsilon_{\lambda} : \lambda \lambda^{\dagger} \Rightarrow I_1,$$

consisting of:

\* for each object  $A$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc} & (1 \times A)(1 \times A) & \\ \tilde{\pi}_A^1 \pi_{1 \times A}^{1 \times A} \swarrow & \downarrow & \searrow \tilde{\pi}_A^1 \pi_{1 \times A}^{1 \times A} \\ A & \epsilon_{\lambda} & A \\ \uparrow 1 & & \downarrow 1 \\ & A & \end{array}$$

where  $\epsilon_{\lambda} := \epsilon_{\lambda A} = \tilde{\pi}_A^1 \pi_{1 \times A}^{1 \times A}$ ,

\* and, for each pair  $A, B \in \text{Span}(\mathcal{B})$ , an identity modification:

$$m_{\epsilon_{\lambda}} : (\epsilon_{\lambda A})^* \lambda \lambda^{\dagger} \Rightarrow I_1 (\epsilon_{\lambda B})_*,$$

consisting of, for each span, an equation of maps of spans,

$$(1, (\epsilon_{\lambda} * 1)(1 * \lambda_{\mu})(\lambda_{\mu} * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi(\lambda_{\mu} * 1)) = (1, I_1(1 * \epsilon_{\lambda}), 1)$$

– a trimodification:

$$\epsilon_{\lambda}^{\dagger} : I_1 \Rightarrow \lambda_{\mu} \lambda_{\mu}^{\dagger},$$

consisting of:

\* for each object  $A$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc} & A & \\ \uparrow 1 & & \downarrow 1 \\ A & \epsilon_{\lambda} & A \\ \tilde{\pi}_A^1 \pi_{1 \times A}^{1 \times A} \swarrow & \downarrow & \searrow \tilde{\pi}_A^1 \pi_{1 \times A}^{1 \times A} \\ & (1 \times A)(1 \times A) & \end{array}$$

where  $\epsilon_{\dot{\lambda}} := \epsilon_{\dot{\lambda}A}$  is the unique 1-cell in  $\mathcal{B}$  such that:

$$\pi_{1 \times A}^{1 \times A} \cdot \mu_{\dot{\lambda}} = \tilde{\pi}_{1 \times A}^A \quad \text{and} \quad \kappa_{1 \times A}^{1,1} \cdot \mu_{\dot{\lambda}} = 1,$$

\* and, for each pair  $A, B \in \text{Span}(\mathcal{B})$ , an identity modification:

$$m_{\epsilon_{\dot{\lambda}}} : (\epsilon_{\dot{\lambda}A})^* I_1 \Rightarrow (\epsilon_{\dot{\lambda}B})_* \lambda \dot{\lambda},$$

consisting of, for each span, an equation of maps of spans,

$$(1, (\epsilon_{\dot{\lambda}A} * 1) I_1, 1) = (1, (1 * \lambda_{\epsilon})(\lambda_{\epsilon} * 1)(1 * \epsilon_{\dot{\lambda}}), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi(\lambda_{\epsilon} * 1)(1 * \epsilon_{\dot{\lambda}B})),$$

– an identity counit perturbation:

$$\epsilon_{\epsilon_{\dot{\lambda}}} : \epsilon_{\dot{\lambda}} \epsilon_{\dot{\lambda}} \Rightarrow 1_{I_1}$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans:

$$\epsilon_{\epsilon_{\dot{\lambda}A}} : \epsilon_{\dot{\lambda}A} \epsilon_{\dot{\lambda}A} \Rightarrow 1_{I_1 A}$$

– and, an identity unit perturbation:

$$\eta_{\epsilon_{\dot{\lambda}}} : 1_{\lambda \dot{\lambda}} \Rightarrow \mu_{\dot{\lambda}} \mu_{\dot{\lambda}}$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans:

$$\eta_{\epsilon_{\dot{\lambda}A}} : 1_{\lambda \dot{\lambda} \cdot A} \Rightarrow \epsilon_{\dot{\lambda}A} \epsilon_{\dot{\lambda}A}$$

• a strict adjoint equivalence  $(\eta_{\dot{\lambda}}, \eta_{\dot{\lambda}}, \epsilon_{\eta_{\dot{\lambda}}}, \eta_{\eta_{\dot{\lambda}}})$ , consisting of:

– a trimodification:

$$\eta_{\dot{\lambda}} : \dot{\lambda} \dot{\lambda} \Rightarrow 1_{\otimes(I \times 1)}$$

consisting of:

\* for each object  $A$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccccc} & & (1 \times A)(1 \times A) & & \\ & \swarrow \pi_{1 \times A}^{1 \times A} & \downarrow \eta_{\dot{\lambda}} & \searrow \pi_{1 \times A}^{1 \times A} & \\ 1 \times A & \xLeftrightarrow{\quad} & & \xLeftrightarrow{\quad} & 1 \times A \\ & \nwarrow 1 & \downarrow & \nearrow 1 & \\ & & 1 \times A & & \end{array}$$

where  $\eta_{\dot{\lambda}} := \eta_{\dot{\lambda}A} = \pi_{1 \times A}^{1 \times A}$ ,

\* and, for each pair of objects  $A, B \in \text{Span}(\mathcal{B})$ , an identity modification:

$$m_{\eta_{\dot{\lambda}}} : (\eta_{\dot{\lambda}A})^* \dot{\lambda} \dot{\lambda} \Rightarrow (\eta_{\dot{\lambda}B})_* I_{\otimes(I \times 1)},$$

consisting of, for each span, an equation of maps of spans:

$$(1, (\eta_{\dot{\lambda}} * 1)(1 * \lambda_{\mu})(\lambda_{\mu} * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi(\lambda_{\mu} * 1)) = (1, I(1 * \eta_{\dot{\lambda}}), 1),$$

– a trimodification:

$$\eta_{\dot{\lambda}} : 1_{\otimes(I \times 1)} \Rightarrow \dot{\lambda} \dot{\lambda},$$

consisting of:

\* for each object  $A$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccccc}
 & & 1 \times A & & \\
 & \swarrow 1 & \downarrow \eta_\lambda & \searrow 1 & \\
 1 \times A & \xrightarrow{\quad} & & \xrightarrow{\quad} & 1 \times A \\
 & \nwarrow \pi_{1 \times A}^{1 \times A} & \downarrow & \nearrow \pi_{1 \times A}^{1 \times A} & \\
 & & (1 \times A)(1 \times A) & & 
 \end{array}$$

where  $\eta_\lambda := \eta_{\lambda A}$  is the unique 1-cell in  $\mathcal{B}$  such that

$$\pi_{1 \times A}^{1 \times A} \cdot \eta_\lambda = 1 \quad \text{and} \quad \kappa_{1 \times A}^{1,1} \cdot \eta_\lambda = 1,$$

\* and, for each pair of objects  $A, B \in \text{Span}(\mathcal{B})$ , an identity modification:

$$m_{\eta_\lambda} : (\eta_{\lambda A})_* I_{\otimes(I \times 1)} \Rightarrow (\eta_{\lambda B})^* \lambda^* \lambda,$$

consisting of, for each span, an equation of maps of spans:

$$(1, I(\eta_\lambda * 1), 1) = (1, (1 * \eta_\lambda)(1 * \lambda_\mu)(\lambda_\mu * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot (\lambda_\mu * 1))$$

consisting of, for each span, an equation of maps of spans,

– an identity counit perturbation:

$$\epsilon_{\eta_\lambda} : \eta_\lambda \eta_\lambda \Rightarrow 1_{\lambda^* \lambda}$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans,

– an identity unit perturbation:

$$\eta_{\eta_\lambda} : 1_{\otimes(I \times 1)} \Rightarrow \eta_\lambda^* \eta_\lambda$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans,

• an identity perturbation:

$$\Phi_\lambda : (1, \mathbf{l}(1 * \eta_\lambda)(\eta_\lambda^* * 1)\mathbf{r}^{-1}, \kappa^{-1} \cdot (1 * \eta_\lambda)(\eta_\lambda^* * 1)\mathbf{r}^{-1}) \Rightarrow (1, 1, 1)$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans,

• and, for each triple of objects  $A, B, C \in \text{Span}(\mathcal{B})$ , an identity isomorphism of maps of spans:

$$\Psi_\lambda : (\kappa \cdot (\eta_\lambda * 1)(1 * \eta_\lambda^*)\mathbf{l}^{-1}, \mathbf{r}(\eta_\lambda * 1)(1 * \eta_\lambda^*)\mathbf{l}^{-1}, 1) \Rightarrow (1, 1, 1).$$

*Proof.* We check that  $\lambda$  and  $\lambda^*$  are tritransformations, that  $\epsilon_\lambda$  and  $\eta_\lambda$  are adjoint equivalences, and that  $\Phi_\lambda$  and  $\Psi_\lambda$  are invertible perturbations. Finally, we verify the axioms of a biadjoint biequivalence.

We first check that  $\lambda$  is a tritransformation and the proof for  $\lambda^*$  follows similarly. We need to verify that  $(\lambda_\mu, \lambda_{\mu^*}, \lambda_\epsilon, \lambda_\eta)$  is an adjoint equivalence of strong transformations and modifications.

Naturality for  $\lambda_{\mu, \bar{R}}$  is a straightforward calculation to showing that, for each map of maps of spans  $\sigma_R$ , the equation:

$$\lambda_{\mu, g_R}((1 \times \sigma_R) * 1_{\lambda_{\mu, \bar{R}}}) = (1_{\lambda_{\mu, R}} * (1 * \sigma_R))\lambda_{\mu, f_R}$$

holds.

Since the monoidal product is locally strict, the transformation axioms simplify. One is a simple calculation and the other is immediate. We have shown that  $\lambda_\mu$  is a strong transformation. Similarly,  $\lambda_{\mu^*}$  is a strong transformation.

The 2-cell equation:

$$(p \cdot (\pi \cdot \lambda_{\epsilon_R}))(\kappa_A^{p, \tilde{\pi}_A^{-1}} \cdot \lambda_\mu \lambda_{\mu^*}) = (\kappa_A^{p, \tilde{\pi}_A^{-1}} \cdot 1)(\tilde{\pi}_A^{-1} \cdot (\pi \cdot \lambda_{\epsilon_R}))$$

allows us to apply the universal property in defining the component isomorphisms of spans of the counit. A similar equation gives the unit isomorphism. The modification axiom for  $\lambda_\epsilon$  is the following equation of 2-cells:

$$1_{f_R}(\lambda_{\epsilon_R} 1_{1*f_R}) = (1_{1*f_R} \lambda_{\epsilon_R})(\lambda_{\mu_{f_R}} 1_{\lambda_{\mu^*R}})(1_{\lambda_{\mu^*R}} \lambda_{\mu^*f_R})$$

which is easily verified by definitions. A similar modification axiom can be checked for  $\lambda_\eta$ .

The transformations and modifications form an adjoint equivalence in a strict 2-category and the axioms reduce to the equations:

$$(1_{\lambda_{\mu^*}} \lambda_\epsilon)(\lambda_\eta 1_{\lambda_{\mu^*}}) = 1_{\lambda_{\mu^*}}$$

and

$$(\lambda_\eta 1_{\lambda_{\mu^*}})(1_{\lambda_{\mu^*}} \lambda_\epsilon) = 1_{\lambda_{\mu^*}},$$

which are verified by simple calculations.

The modification axioms are immediate for the collections of identity cells  $\lambda_\Pi$  and  $\lambda_M$ .

The tritransformation axioms are immediate since all modifications cells from  $\text{Span}(\mathcal{B})$ , the monoidal product, and  $\lambda$  are identities. It follows that  $\lambda$  and similarly  $\lambda^*$  are tritransformations.

Next we check that  $\epsilon_\lambda$  and  $\epsilon_\lambda^*$  are trimodifications and the 1-cells of an adjoint equivalence. Similar results will hold for  $\eta_\lambda$  and  $\eta_\lambda^*$ .

The modification axiom is immediate since the 2-cells  $m_{\epsilon_\lambda R}$  are identities. The trimodification axioms are also immediately satisfied since these 2-cells and the modification components  $\lambda_\Pi$ ,  $\lambda_M$ , and the analogous modifications for  $\lambda^*$  and  $I_1$  are all identities. It follows that  $\epsilon_\lambda$ , and similarly  $\epsilon_\lambda^*$  are trimodifications.

The counit and unit perturbations  $\epsilon_{\epsilon_\lambda}$  and  $\eta_{\epsilon_\lambda}$  are each identities and thus trivially satisfy the perturbation axioms. Further the adjoint equivalence axioms are immediately satisfied. It follows that  $(\epsilon_\lambda, \epsilon_\lambda^*, \epsilon_{\epsilon_\lambda}, \eta_{\epsilon_\lambda})$ , and similarly  $(\eta_\lambda, \eta_\lambda^*, \epsilon_{\eta_\lambda}, \eta_{\eta_\lambda})$ , are adjoint equivalences.

Finally, the axioms of a biadjoint biequivalence will be satisfied since the perturbations  $\Phi_\lambda$  and  $\Psi_\lambda$  are identities and the modifications of  $\text{Span}(\mathcal{B})$  are all identities. To check these equations precisely we need to specify the tricategory structure on  $\text{Tricat}(\text{Span}(\mathcal{B}), \text{Span}(\mathcal{B}))$ , but we do not include these details here.  $\square$

## Monoidal Right Unitor

The monoidal right unitor is a biadjoint biequivalence:

$$\rho: \otimes(1 \times I) \Rightarrow 1$$

consisting of transformations, adjoint equivalences of modifications and perturbations, and coherence perturbations consisting of isomorphisms of maps of spans.

**Proposition 46.** *There is a biadjoint biequivalence  $(\rho, \rho^*, \epsilon_\rho, \eta_\rho, \Phi_\rho, \Psi_\rho)$  consisting of:*

- *a tritransformation:*

$$\rho: \otimes(1 \times I) \Rightarrow 1,$$

*consisting of:*

- *for each object  $A \in \text{Span}(\mathcal{B})$ , the span  $\rho_A$ :*

$$\begin{array}{ccc} & A \times 1 & \\ \swarrow \pi_A^1 & & \searrow 1 \\ A & & A \times 1 \end{array}$$

- *for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , an adjoint equivalence:*

$$(\rho_\mu, \rho_{\mu^*}, \rho_\epsilon, \rho_\eta): \text{hom}_{\text{Span}}(1, \rho_B)(\otimes(1 \times I)) \Rightarrow \text{hom}_{\text{Span}}(\rho_A, 1)1,$$

*in a strict 2-category of transformations and modifications, consisting of:*

- \* *a strong transformation:*

$$\rho_{\mu_{A,B}}: \text{hom}_{\text{Span}}(1, \rho_B)(\otimes(1 \times I)) \Rightarrow \text{hom}_{\text{Span}}(\rho_A, 1)1,$$

*consisting of:*

· for each span  $R$ , a map of spans:

$$\begin{array}{ccccc}
 & & (B \times 1)(R \times 1) & & \\
 & \swarrow \tilde{\pi}_B^1 \pi_{B \times 1}^{R \times 1} & \downarrow & \searrow (1 \times p) \pi_{R \times 1}^{B \times 1} & \\
 B & & \rho_\mu & & A \times 1 \\
 & \nwarrow q \pi_R^{A \times 1} & \downarrow & \nearrow \pi_{A \times 1}^R & \\
 & & R(A \times 1) & & 
 \end{array}$$

$\tilde{\pi}_B^1 \cdot \kappa^{-1}$  (between  $B$  and  $A \times 1$ )  
 $\rho_\mu$  (vertical arrow from  $(B \times 1)(R \times 1)$  to  $R(A \times 1)$ )

where  $\kappa := \kappa_{B \times 1}^{(B \times 1)(R \times 1)}$  and  $\rho_\mu := \rho_{\mu_R}$  is the unique 1-cell satisfying:

$$\pi_{A \times 1}^R \rho_{\mu_R} = (p \times 1) \pi_{R \times 1}^{B \times 1} \quad \pi_R^{A \times 1} \rho_{\mu_R} = \tilde{\pi}_R^1 \pi_{R \times 1}^{B \times 1},$$

and

$$\kappa_A^{R(A \times 1)} \cdot \rho_{\mu_R} = 1,$$

· for each pair of spans  $R, \bar{R}$ , a natural isomorphism:

$$\rho_{\mu_R, \bar{R}}: (\rho_{\mu_{\bar{R}}})_* \text{hom}_{\text{Span}}(1, \rho_B)(\otimes(1 \times I)) \Rightarrow (\rho_{\mu_R})^* \text{hom}_{\text{Span}}(\rho_A, 1)1,$$

consisting of, for each map of spans  $f_R$ , an isomorphism of maps of spans:

$$\rho_{\mu_{f_R}}: ((\alpha_R \times 1) \cdot \pi, \rho_{\mu_{\bar{R}}}((f_R \times 1) * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot ((f_R \times 1) * 1)) \Rightarrow (1, (1 * f_R) \rho_{\mu_R}, (\beta_R \cdot \pi \rho_{\mu_R})(\tilde{\pi} \cdot \kappa^{-1})),$$

consisting of the unique 2-cell:

$$\rho_{\mu_{f_R}}: \rho_{\mu_{\bar{R}}}((f_R \times 1) * 1) \Rightarrow (1 * f_R) \rho_{\mu_R}$$

in  $\mathcal{B}$  such that:

$$\pi_{A \times 1}^{\bar{R}} \cdot \rho_{\mu_{f_R}} = (\alpha_R^{-1} \times 1) \cdot \pi_{R \times 1}^{B \times 1} \quad \text{and} \quad \pi_{\bar{R}}^{A \times 1} \cdot \rho_{\mu_{f_R}} = 1,$$

\* a strong transformation:

$$\rho_{\mu^* A, B}: \text{hom}_{\text{Span}}(\rho_A, 1)1 \Rightarrow \text{hom}_{\text{Span}}(1, \rho_B)(\otimes(1 \times I)),$$

consisting of:

· for each span  $R$ , a map of spans:

$$\begin{array}{ccccc}
 & & R(A \times 1) & & \\
 & \swarrow q \pi_R^{A \times 1} & \downarrow & \searrow \pi_{A \times 1}^R & \\
 B & & \rho_{\mu^*} & & A \times 1 \\
 & \nwarrow \tilde{\pi}_B^1 \pi_{B \times 1}^{R \times 1} & \downarrow & \nearrow (p \times 1) \pi_{R \times 1}^{B \times 1} & \\
 & & (B \times 1)(R \times 1) & & 
 \end{array}$$

$\tilde{\pi} \cdot \kappa$  (between  $B$  and  $A \times 1$ )

where  $\tilde{\pi} \cdot \kappa := \tilde{\pi}_{A \times 1}^A \cdot \kappa_A^{R(A \times 1)}$  and  $\rho_{\mu^*} := \rho_{\mu^* R}$  is the unique 1-cell satisfying:

$$\pi_{R \times 1}^{B \times 1} \rho_{\mu^* R} = \tilde{\pi}_{R \times 1}^R \pi_R^{A \times 1} \quad \pi_{B \times 1}^{R \times 1} \rho_{\mu^* R} = (q \times 1) \tilde{\pi}_{R \times 1}^R \pi_R^{A \times 1}$$

and

$$\kappa_{B \times 1}^{(B \times 1)(R \times 1)} \cdot \rho_{\mu^* R} = 1,$$

· for each pair of spans  $R, \bar{R}$ , a natural isomorphism:

$$\rho_{\mu \cdot R, \bar{R}}: (\rho_{\mu \cdot \bar{R}})_* \text{hom}_{\text{Span}}(\rho_A, 1)1 \Rightarrow (\rho_{\mu \cdot R})^* \text{hom}_{\text{Span}}(1, \rho_B)(\otimes(1 \times I)),$$

consisting of, for each map of spans  $f_R$ , an isomorphism of maps of spans:

$$\rho_{\mu \cdot f_R}: (\tilde{\pi} \cdot \kappa \cdot (1 * f_R), \rho_{\mu \cdot \bar{R}}(1 * f_R), \beta_R \cdot \pi) \Rightarrow (((\alpha_R \times 1) \cdot \pi \rho_{\mu \cdot R})(\tilde{\pi} \cdot \kappa), ((f_R \times 1) * 1) \rho_{\mu \cdot R}, 1)$$

consisting of the unique 2-cell:

$$\rho_{\mu \cdot f_R}: \rho_{\mu \cdot \bar{R}}(1 * f_R) \Rightarrow ((f_R \times 1) * 1) \rho_{\mu \cdot R}$$

in  $\mathcal{B}$  such that:

$$\pi_{\bar{R} \times 1}^{B \times 1} \cdot \rho_{\mu \cdot f_R} = 1 \quad \text{and} \quad \pi_{B \times 1}^{\bar{R} \times 1} \cdot \rho_{\mu \cdot f_R} = \tilde{\pi}_{B \times 1}^B \beta_R^{-1} \cdot \pi_R^{A \times 1},$$

\* an invertible counit modification:

$$\rho_\epsilon: \rho_\mu \rho_{\mu^*} \Rightarrow 1$$

consisting of, for each span  $R$ , an isomorphism of maps of spans:

$$\rho_{\epsilon R}: (\tilde{\pi} \cdot \kappa, \rho_\mu \rho_{\mu^*}, \tilde{\pi}_B^1 \cdot \kappa^{-1} \cdot \rho_{\mu^*}) \Rightarrow (1, 1, 1)$$

defined by the unique 2-cell  $\rho_{\epsilon R}$  in  $\mathcal{B}$  such that:

$$\pi_{A \times 1}^R \cdot \rho_{\epsilon R} = \tilde{\pi}_{A \times 1}^A \cdot \kappa_A^{p, \tilde{\pi}_A^1 - 1} \quad \text{and} \quad \pi_R^{A \times 1} \cdot \rho_{\epsilon R} = 1,$$

\* an invertible unit modification:

$$\rho_\eta: 1 \Rightarrow \rho_{\mu^*} \rho_\mu$$

consisting of, for each span  $R$ , an isomorphism of maps of spans:

$$\rho_{\eta R}: (1, 1, 1) \Rightarrow (\tilde{\pi} \cdot \kappa \cdot \rho_\mu, \rho_{\mu^*} \rho_\mu, \tilde{\pi} \cdot \kappa^{-1})$$

defined by the unique 2-cell  $\rho_{\eta R}$  in  $\mathcal{B}$  such that:

$$\pi_{R \times 1}^{B \times 1} \cdot \rho_{\eta R} = 1 \quad \text{and} \quad \pi_{B \times 1}^{R \times 1} \cdot \rho_{\eta R} = \kappa_{B \times 1}^{1, q \times 1 - 1},$$

– an identity modification with component equations of maps of spans:

$$(1, \chi(\rho_{\mu R} * 1)(1 * \rho_{\mu S}), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi) = (1, \rho_{\mu SR}(\chi * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot (\chi * 1)),$$

– an identity modification with component equations of maps of spans:

$$(1, \rho_{\mu A} \iota \mathbf{r}^{-1}, \tilde{\pi} \cdot \kappa^{-1} \cdot \iota \mathbf{r}^{-1}) = (1, \iota \mathbf{l}^{-1}, 1),$$

• a tritransformation:

$$\rho: 1 \Rightarrow \otimes(1 \times I),$$

consisting of:

– for each object  $A \in \text{Span}(\mathcal{B})$ , the span  $\rho_A$ :

$$\begin{array}{ccc} & A \times 1 & \\ \swarrow 1 & & \searrow \tilde{\pi}_A^1 \\ A \times 1 & & A \end{array}$$

– for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , an adjoint equivalence:

$$(\rho_\mu, \rho_{\mu^*}, \rho_\epsilon, \rho_\eta): \text{hom}_{\text{Span}}(1, \rho_B)1 \Rightarrow \text{hom}_{\text{Span}}(\rho_A, 1)(\otimes(1 \times I)),$$

in a strict 2-category of transformations and modifications, consisting of:



\* a strong transformation:

$$\rho_{\mu_{A,B}}: \text{hom}_{\text{Span}}(1, \rho_B)1 \Rightarrow \text{hom}_{\text{Span}}(\rho_A, 1)(\otimes(1 \times I)),$$

consisting of:

· for each span  $R$ , a map of spans:

$$\begin{array}{ccc}
 & (B \times 1)R & \\
 \pi_{B \times 1}^R \swarrow & & \searrow p\pi_{R \times 1}^{B \times 1} \\
 B \times 1 & \xrightarrow{\tilde{\pi} \cdot \kappa^{-1}} & A \\
 (q \times 1)\pi_{R \times 1}^{A \times 1} \swarrow & \rho_{\mu} & \nearrow \tilde{\pi}_A^1 \pi_{A \times 1}^{R \times 1} \\
 & (R \times 1)(A \times 1) &
 \end{array}$$

where  $\tilde{\pi} \cdot \kappa^{-1} := \tilde{\pi}_{B \times 1}^B \cdot \kappa_{B \times 1}^{B \times 1, q}{}^{-1}$  and  $\rho_{\mu} := \rho_{\mu_R}$  is the unique 1-cell satisfying:

$$\pi_{A \times 1}^{R \times 1} \rho_{\mu_R} = (p \times 1) \tilde{\pi}_{R \times 1}^R \pi_R^{B \times 1} \quad \pi_{R \times 1}^{A \times 1} \rho_{\mu_R} = \tilde{\pi}_{R \times 1}^R \pi_R^{B \times 1}$$

and

$$\kappa_{A \times 1}^{p \times 1, 1} \cdot \rho_{\mu_R} = 1,$$

· for each pair of spans  $R, \bar{R}$ , a natural isomorphism:

$$\rho_{\mu_{R, \bar{R}}}: \left( \rho_{\mu_{\bar{R}}} \right)_* \text{hom}_{\text{Span}}(1, \rho_B)1 \Rightarrow \left( \rho_{\mu_R} \right)^* \text{hom}_{\text{Span}}(\rho_A, 1)(\otimes(1 \times I)),$$

consisting of, for each map of spans  $f_R$ , an isomorphism of maps of spans:

$$\rho_{\mu_{f_R}}: (\alpha_R \cdot \pi, \rho_{\mu_{\bar{R}}}(f_R * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot (f_R * 1)) \Rightarrow (1, (1 * (f_R \times 1))\rho_{\mu_R}, ((\beta_R \times 1) \cdot \pi \rho_{\mu_R})(\tilde{\pi} \cdot \kappa^{-1})),$$

consisting of the unique 2-cell:

$$\rho_{\mu_{f_R}}: \rho_{\mu_{\bar{R}}}(f_R * 1) \Rightarrow (1 * (f_R \times 1))\rho_{\mu_R}$$

in  $\mathcal{B}$  such that:

$$\pi_{\bar{R} \times 1}^{A \times 1} \cdot \rho_{\mu_{f_R}} = 1 \quad \text{and} \quad \pi_{A \times 1}^{\bar{R} \times 1} \cdot \rho_{\mu_{f_R}} = (\alpha_R^{-1} \times 1) \cdot \tilde{\pi}_{R \times 1}^R \pi_R^{B \times 1},$$

\* a strong transformation:

$$\rho_{\mu_{A,B}}: \text{hom}_{\text{Span}}(\rho_A, 1)(\otimes(1 \times I)) \Rightarrow \text{hom}_{\text{Span}}(1, \rho_B)1,$$

consisting of:

· for each span  $R$ , a map of spans:

$$\begin{array}{ccc}
 & (R \times 1)(A \times 1) & \\
 (q \times 1)\pi_{R \times 1}^{A \times 1} \swarrow & & \searrow \tilde{\pi}_A^1 \pi_{A \times 1}^{R \times 1} \\
 B \times 1 & \xrightarrow{\rho_{\mu}} & A \\
 \pi_{B \times 1}^R \swarrow & & \searrow p\pi_{R \times 1}^{B \times 1} \\
 & (B \times 1)R &
 \end{array}$$

where  $\kappa := \kappa_{A \times 1}^{p \times 1, 1}$  and  $\rho_{\mu \cdot R} := \rho_{\mu \cdot R}$  is the unique 1-cell satisfying:

$$\pi_R^{B \times 1} \rho_{\mu \cdot R} = \tilde{\pi}_R^1 \pi_{R \times 1}^{A \times 1} \quad \pi_{B \times 1}^R \rho_{\mu \cdot R} = (q \times 1) \pi_{R \times 1}^{A \times 1}$$

and

$$\kappa_B^{\tilde{\pi}_R^1, q} \cdot \rho_{\mu \cdot R} = 1,$$

· for each pair of spans  $R, \bar{R}$ , a natural isomorphism:

$$\rho_{\mu \cdot R, \bar{R}} : \left( \rho_{\mu \cdot \bar{R}} \right)_* \text{hom}_{\text{Span}}(1, \rho_{\bar{B}})(\otimes(1 \times I)) \Rightarrow \left( \rho_{\mu \cdot R} \right)^* \text{hom}_{\text{Span}}(\rho_A, 1)1,$$

consisting of, for each map of spans  $f_R$ , an isomorphism of maps of spans:

$$\rho_{\mu \cdot f_R} : (\tilde{\pi} \cdot \kappa \cdot (1 * (f_R \times 1)), \rho_{\mu \cdot \bar{R}}(1 * (f_R \times 1)), (\beta_R \times 1) \cdot \pi) \Rightarrow$$

$$(((\alpha_R \times 1) \cdot \pi \rho_{\mu \cdot R})(\tilde{\pi} \cdot \kappa), (f_R * 1) \rho_{\mu \cdot R}, 1),$$

consisting of the unique 2-cell:

$$\rho_{\mu \cdot f_R} : \rho_{\mu \cdot \bar{R}}(1 * (f_R \times 1)) \Rightarrow (f_R * 1) \rho_{\mu \cdot R}$$

in  $\mathcal{B}$  such that:

$$\pi_R^{B \times 1} \cdot \rho_{\mu \cdot f_R} = 1 \quad \text{and} \quad \pi_{B \times 1}^{\bar{R}} \cdot \rho_{\mu \cdot f_R} = (\beta_R^{-1} \times 1) \cdot \pi_{R \times 1}^{A \times 1},$$

\* an invertible counit modification:

$$\rho_{\epsilon} : \rho_{\mu} \rho_{\mu} \Rightarrow 1$$

consisting of, for each span  $R$ , an isomorphism of maps of spans:

$$\rho_{\epsilon R} : (\tilde{\pi} \cdot \kappa, \rho_{\mu} \rho_{\mu}, \tilde{\pi} \cdot \kappa^{-1} \cdot \rho_{\mu}) \Rightarrow (1, 1, 1)$$

defined by the unique 2-cell  $\rho_{\epsilon R}$  in  $\mathcal{B}$  such that:

$$\pi_{A \times 1}^{R \times 1} \cdot \rho_{\epsilon R} = \kappa_{A \times 1}^{p \times 1, 1-1} \quad \text{and} \quad \pi_{R \times 1}^{A \times 1} \cdot \rho_{\epsilon R} = 1,$$

\* an invertible unit modification:

$$\rho_{\eta} : 1 \Rightarrow \rho_{\mu} \rho_{\mu}$$

consisting of, for each span  $R$ , an isomorphism of maps of spans:

$$\rho_{\eta R} : (1, 1, 1) \Rightarrow (\tilde{\pi} \cdot \kappa \cdot \rho_{\mu}, \rho_{\mu} \rho_{\mu}, \tilde{\pi} \cdot \kappa^{-1})$$

defined by the unique 2-cell  $\rho_{\eta R}$  in  $\mathcal{B}$  such that:

$$\pi_R^{B \times 1} \cdot \rho_{\eta R} = 1 \quad \text{and} \quad \pi_{B \times 1}^R \cdot \rho_{\eta R} = \tilde{\pi}_{B \times 1}^B \cdot \kappa_B^{\tilde{\pi}_B^1, q^{-1}},$$

– an identity modification with component equations of maps of spans:

$$(1, \chi(\rho_{\mu R} * 1)(1 * \rho_{\mu S}), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi) = (1, \rho_{\mu S R}(\chi * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot (\chi * 1)),$$

– an identity modification with component equations of maps of spans:

$$(1, \rho_{\mu A} \iota \mathbf{r}^{-1}, \tilde{\pi} \cdot \kappa^{-1} \cdot \iota \mathbf{r}^{-1}) = (1, \iota \mathbf{l}^{-1}, 1),$$

• a strict adjoint equivalence  $(\mu_{\rho}, \mu_{\rho}, \epsilon_{\mu_{\rho}}, \eta_{\mu_{\rho}})$ : consisting of:

– a trimodification:

$$\mu_{\rho} : \rho \rho \Rightarrow I_1,$$

consisting of:

\* for each object  $A$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc}
 & (A \times 1)(A \times 1) & \\
 \swarrow \tilde{\pi}_A^1 \pi_{A \times 1}^{A \times 1} & \downarrow & \searrow \tilde{\pi}_A^1 \pi_{A \times 1}^{A \times 1} \\
 A & \mu_\rho & A \\
 \nwarrow 1 & & \nearrow 1 \\
 & A &
 \end{array}$$

where  $\mu_\rho := \mu_{\rho A} = \tilde{\pi}_A^1 \pi_{A \times 1}^{A \times 1}$ ,

\* and, for each pair  $A, B \in \text{Span}(\mathcal{B})$ , an identity modification:

$$m_{\mu_\rho} : (1, (\mu_\rho * 1)(1 * \rho_\mu)(\rho_\mu * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi(\rho_\mu * 1)) \Rightarrow (1, I_1(1 * \mu_\rho), 1),$$

consisting of, for each span, an equation of maps of spans,

– a trimodification:

$$\mu'_\rho : I_1 \Rightarrow \rho \rho,$$

consisting of:

\* for each object  $A$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc}
 & A & \\
 \swarrow 1 & \downarrow & \searrow 1 \\
 A & \mu'_\rho & A \\
 \nwarrow \tilde{\pi}_A^1 \pi_{A \times 1}^{A \times 1} & & \nearrow \tilde{\pi}_A^1 \pi_{A \times 1}^{A \times 1} \\
 & (A \times 1)(A \times 1) &
 \end{array}$$

where  $\mu'_\rho$  is the unique 1-cell in  $\mathcal{B}$  such that

$$\pi_{A \times 1}^{A \times 1} \cdot \mu'_\rho = \tilde{\pi}_{A \times 1}^A \quad \text{and} \quad \kappa_{A \times 1}^{(A \times 1)(A \times 1)} \cdot \mu'_\rho = 1,$$

\* and, for each pair  $A, B \in \text{Span}(\mathcal{B})$ , an identity modification:

$$m_{\mu'_\rho} : (1, (\mu'_\rho * 1)I, 1) \Rightarrow (1, (1 * \rho_\mu)(\rho_\mu * 1)(1 * \mu'_\rho), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi(\rho_\mu * 1)(1 * \mu'_\rho)),$$

consisting of, for each span, an equation of maps of spans,

– an identity counit perturbation:

$$\epsilon_{\mu_\rho} : \mu_\rho \mu'_\rho \Rightarrow 1_{I_1}$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans,

– and, an identity unit perturbation:

$$\eta_{\mu_\rho} : 1_{\rho \rho} \Rightarrow \mu'_\rho \mu_\rho$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans,

• a strict adjoint equivalence  $(\eta_\rho, \eta'_\rho, \epsilon_{\eta_\rho}, \eta_{\eta_\rho})$ , consisting of:

– a trimodification:

$$\eta_\rho : \rho \rho \Rightarrow 1_{1 \times I}$$

\* for each object  $A \in \text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc}
 & (A \times 1)(A \times 1) & \\
 \pi_{A \times 1}^{A \times 1} \swarrow & \downarrow \eta_\rho & \searrow \pi_{A \times 1}^{A \times 1} \\
 A \times 1 & & A \times 1 \\
 \uparrow 1 & & \downarrow 1 \\
 & A \times 1 &
 \end{array}$$

where  $\eta_\rho := \eta_{\rho A} = \pi_{A \times 1}^{A \times 1}$ ,

\* and, for each pair of objects  $A, B \in \text{Span}(\mathcal{B})$ , an identity modification:

$$m_{\eta_\rho} : (1, (\eta_\rho * 1)(1 * \rho_\mu)(\rho_\mu * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi(\rho_\mu * 1)) = (1, I(1 * \eta_\rho), 1),$$

consisting of, for each span, an equation of maps of spans,

– a trimodification:

$$\eta_\rho^\cdot : 1_{\otimes(1 \times I)} \Rightarrow \rho^\cdot \rho,$$

consisting of:

\* for each object  $A \in \text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc}
 & A \times 1 & \\
 1 \swarrow & \downarrow \eta_\rho^\cdot & \searrow 1 \\
 A \times 1 & & A \times 1 \\
 \pi_{A \times 1}^{A \times 1} \swarrow & & \searrow \pi_{A \times 1}^{A \times 1} \\
 & (A \times 1)(A \times 1) &
 \end{array}$$

where  $\eta_\rho^\cdot := \eta_{\rho^\cdot A}$  is the unique 1-cell in  $\mathcal{B}$  such that

$$\pi_{A \times 1}^{A \times 1} \cdot \eta_\rho^\cdot = 1 \quad \text{and} \quad \kappa_{A \times 1}^{(A \times 1)(A \times 1)} \cdot \eta_\rho^\cdot = 1,$$

\* and, for each pair of objects  $A, B \in \text{Span}(\mathcal{B})$ , an identity modification:

$$m_{\eta_\rho^\cdot} : (1, (\eta_\rho^\cdot * 1)I, 1) = (1, (1 * \rho_\mu)(\rho_\mu * 1)(1 * \eta_\rho^\cdot), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi(\rho_\mu * 1)(1 * \eta_\rho^\cdot))$$

consisting of, for each span, an equation of maps of spans,

– an identity counit perturbation:

$$\epsilon_{\eta_\rho} : \eta_\rho \eta_\rho^\cdot \Rightarrow 1_{\rho^\cdot \rho}$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans,

– and, an identity unit perturbation:

$$\eta_{\eta_\rho} : 1_{\otimes(1 \times I)} \Rightarrow \eta_\rho^\cdot \eta_\rho$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans,

• an identity perturbation:

$$\Phi_\rho : (1, I(1 * \mu_\rho)(\eta_\rho^\cdot * 1)\mathbf{r}^{-1}, \kappa^{-1} \cdot (1 * \mu_\rho)(\eta_\rho^\cdot * 1)\mathbf{r}^{-1}) \Rightarrow (1, 1, 1),$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans,

- an identity perturbation:

$$\Psi_\rho: (\kappa \cdot (\mu_\rho * 1)(1 * \eta_\rho) \mathbf{I}^{-1}, \mathbf{r}(\mu_\rho * 1)(1 * \eta_\rho) \mathbf{I}^{-1}, 1) \Rightarrow (1, 1, 1)$$

consisting of, for each object  $A \in \text{Span}(\mathcal{B})$ , an equation of maps of spans.

*Proof.* The proof is similar to that of the previous proposition for the biadjoint biequivalence  $\lambda$ . We omit the details.  $\square$

## 5.6 Monoidal Adjoint Equivalences

### Monoidal Pentagonator Modification

We define a strict adjoint equivalence:

$$\Pi: (1 \times \alpha)\alpha(\alpha \times 1) \Rightarrow \alpha\alpha$$

in the strict 2-category of tritransformations, trimodifications, and perturbations.

**Proposition 47.** *There is a strict adjoint equivalence  $(\Pi, \Pi^*, 1, 1)$  consisting of:*

- a trimodification

$$\Pi: (1 \times \alpha)\alpha(\alpha \times 1) \Rightarrow \alpha\alpha$$

consisting of:

- for each 4-tuple of objects  $A, B, C, D$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc} (A \times ((B \times C) \times D))(((A \times (B \times C)) \times D)((A \times B) \times C) \times D) & & \\ \swarrow \scriptstyle (1_A \times a_{BCD})\pi_{A \times ((B \times C) \times D)}^{((A \times (B \times C)) \times D)((A \times B) \times C) \times D} & & \searrow \scriptstyle \pi_{((A \times B) \times C) \times D}^{(A \times (B \times C)) \times D} \pi_{((A \times (B \times C)) \times D)((A \times B) \times C) \times D}^{A \times ((B \times C) \times D)} \\ & \Pi & \\ \swarrow \scriptstyle a_{AB(CD)}\pi_{(A \times B) \times (C \times D)}^{(A \times B) \times (C \times D)} & & \searrow \scriptstyle \pi_{((A \times B) \times C) \times D}^{(A \times B) \times (C \times D)} \\ ((A \times B) \times (C \times D))(((A \times B) \times C) \times D) & & \end{array}$$

where  $\Pi := \Pi_{ABCD}$  is the unique 1-cell in  $\mathcal{B}$  such that:

$$\pi_{((A \times B) \times C) \times D}^{(A \times B) \times (C \times D)} \cdot \Pi_{ABCD} = \pi_{((A \times B) \times C) \times D}^{(A \times (B \times C)) \times D} \pi_{((A \times (B \times C)) \times D)((A \times B) \times C) \times D}^{A \times ((B \times C) \times D)},$$

$$\pi_{(A \times B) \times (C \times D)}^{((A \times B) \times C) \times D} \cdot \Pi_{ABCD} = a_{AB(CD)}^{-1} (1_A \times a_{BCD}) \pi_{A \times ((B \times C) \times D)}^{((A \times (B \times C)) \times D)((A \times B) \times C) \times D},$$

and

$$\kappa_{(A \times B) \times (C \times D)}^{1, a} \cdot \Pi_{ABCD} = 1,$$

- and, for each two 4-tuples  $(A, B, C, D), (A', B', C', D')$  of objects in  $\text{Span}(\mathcal{B})$ , an identity modification:

$$m_\Pi: (1, (\Pi * 1)(1 * (1 \times \alpha))((1 * \alpha) * 1)((\alpha \times 1) * 1) * 1), (1 \times (a \cdot \kappa^{-1})) \cdot \pi(1 * \alpha * 1)((\alpha \times 1) * 1 * 1))$$

$$\Rightarrow (1, (1 * \alpha)(\alpha * 1)(1 * \Pi), a \cdot \kappa^{-1} \cdot \pi(\alpha * 1)(1 * \Pi)),$$

consisting of, for each four spans, an equation of maps of spans,

- a trimodification

$$\Pi^*: \alpha\alpha \Rightarrow (1 \times \alpha)\alpha(\alpha \times 1),$$

consisting of:

– for each 4-tuple of objects  $A, B, C, D$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc}
 & ((A \times B) \times (C \times D))((A \times B) \times C) \times D & \\
 a_{AB(CD)} \pi_{((A \times B) \times C) \times D}^{((A \times B) \times C) \times D} \swarrow & \downarrow & \searrow \pi_{((A \times B) \times C) \times D}^{(A \times B) \times (C \times D)} \\
 A \times (B \times (C \times D)) \Downarrow & \Pi^* & \Downarrow ((A \times B) \times C) \times D \\
 (1_A \times a_{BCD}) \pi_{A \times ((B \times C) \times D)}^{((A \times (B \times C)) \times D)((A \times B) \times C) \times D} \swarrow & \downarrow & \searrow \pi_{((A \times B) \times C) \times D}^{(A \times (B \times C)) \times D} \pi_{((A \times (B \times C)) \times D)((A \times B) \times C) \times D}^{A \times ((B \times C) \times D)} \\
 & (A \times ((B \times C) \times D))((A \times (B \times C)) \times D)((A \times B) \times C) \times D &
 \end{array}$$

where and  $\Pi^* := \Pi_{ABCD}^*$  is a 1-cell in  $\mathcal{B}$  such that:

$$\begin{aligned}
 \pi_{((A \times B) \times C) \times D}^{((A \times (B \times C)) \times D) A \times ((B \times C) \times D)} \pi_{((A \times B) \times C) \times D}^{((A \times (B \times C)) \times D)((A \times B) \times C) \times D} \cdot \Pi_{ABCD}^* &= \pi_{((A \times B) \times C) \times D}^{(A \times B) \times (C \times D)}, \\
 \pi_{((A \times B) \times C) \times D}^{((A \times B) \times C) \times D} \pi_{((A \times (B \times C)) \times D)((A \times B) \times C) \times D}^{A \times ((B \times C) \times D)} \cdot \Pi_{ABCD}^* &= (a \times 1) a^{-1} \pi_{((A \times B) \times C) \times D}^{((A \times B) \times C) \times D}, \\
 \pi_{A \times ((B \times C) \times D)}^{((A \times (B \times C)) \times D)((A \times B) \times C) \times D} \cdot \Pi_{ABCD}^* &= (1 \times a^{-1}) a \pi_{(A \times B) \times (C \times D)}^{(A \times B) \times (C \times D)},
 \end{aligned}$$

and

$$\kappa_{A \times ((B \times C) \times D)}^{1, a \pi_{(A \times (B \times C)) \times D}^{(A \times B) \times C} \times D} \cdot \Pi_{ABCD}^* = ((a \times 1) a^{-1}) \cdot \kappa_{(A \times B) \times (C \times D)}^{1, a},$$

– and, for each two 4-tuples  $(A, B, C, D), (A', B', C', D')$  of objects in  $\text{Span}(\mathcal{B})$ , an identity modification:

$$m_{\Pi^*} : (1, (\Pi^* * 1)(1 * \alpha)(\alpha * 1), a \cdot \kappa^{-1} \cdot (\alpha * 1)) \Rightarrow$$

$$(1, (1 * (1 \times \alpha))((1 * \alpha * 1)((\alpha \times 1) * 1 * 1)(1 * \Pi^*), (1 \times a \cdot \kappa^{-1}) \cdot \pi(1 * \alpha * 1)((\alpha \times 1) * 1 * 1)(1 * \Pi^*)),$$

consisting of, for each three spans, an equation of maps of spans,

- and, identity counit and unit perturbations.

*Proof.* Since  $m_{\Pi}$  and the modification components of  $\alpha$  are all identities,  $\Pi$  is a trimodification.

To define  $\Pi^*$  uniquely we need an auxiliary map, which is the unique 1-cell

$$\Pi^0 : ((A \times B) \times (C \times D))((A \times B) \times C) \times D \rightarrow ((A \times (B \times C)) \times D)((A \times B) \times C) \times D$$

in  $\mathcal{B}$  such that:

$$\pi_{((A \times B) \times C) \times D}^{(A \times (B \times C)) \times D} \cdot \Pi^0 = \pi_{((A \times B) \times C) \times D}^{(A \times B) \times (C \times D)}, \quad \pi_{(A \times (B \times C)) \times D}^{((A \times B) \times C) \times D} \cdot \Pi^0 = (a \times 1) a^{-1} \pi_{(A \times B) \times (C \times D)}^{((A \times B) \times C) \times D},$$

and

$$\kappa_{(A \times (B \times C)) \times D}^{1, a \times 1} \cdot \Pi^0 = (a \times 1) a^{-1} \cdot \kappa_{(A \times B) \times (C \times D)}^{((A \times B) \times (C \times D))((A \times B) \times C) \times D}.$$

It is then straightforward to see that  $\Pi^*$  is a modification, and that the pair of modifications define a strict adjoint equivalence.  $\square$

## Monoidal Left Mediator Unit Modification

We define an adjoint equivalence:

$$l : \lambda \alpha \Rightarrow (\lambda \times 1)$$

in the strict 2-category of tritransformations, trimodifications, and perturbations.

**Proposition 48.** *There is an adjoint equivalence  $(l, l^*, 1, \eta_l)$ , consisting of:*

- a trimodification:

$$l : \lambda \alpha \Rightarrow (\lambda \times 1),$$

consisting of:

- for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc}
 & (1 \times (A \times B))((1 \times A) \times B) & \\
 \tilde{\pi}_{A \times B}^1 \pi_{1 \times (A \times B)}^{(1 \times A) \times B} \swarrow & \downarrow l_{AB} & \searrow \pi_{(1 \times A) \times B}^{1 \times (A \times B)} \\
 A \times B & & (1 \times A) \times B \\
 \tilde{\pi}_A^1 \times 1_B \swarrow & & \searrow 1 \\
 & (1 \times A) \times B &
 \end{array}$$

$\kappa^{-1}$  (between  $A \times B$  and  $(1 \times A) \times B$ )  
 $\pi_{(1 \times A) \times B}^{1 \times (A \times B)}$  (between  $(1 \times A) \times B$  and  $(1 \times (A \times B))((1 \times A) \times B)$ )  
 $\tilde{\pi}_{A \times B}^1 \pi_{1 \times (A \times B)}^{(1 \times A) \times B}$  (between  $A \times B$  and  $(1 \times (A \times B))((1 \times A) \times B)$ )

where  $\kappa := \kappa_{1 \times (A \times B)}^{1, a}$  and  $l_{AB} := \pi_{(1 \times A) \times B}^{1 \times (A \times B)}$ ,

- and, for each two pairs  $(A, B), (A', B')$  of objects in  $\text{Span}(\mathcal{B})$ , an identity modification:

$$m_l: (1, (l * 1)(1 * \lambda)(\alpha * 1), (\tilde{\pi} \cdot \kappa^{-1}) \cdot \pi(\alpha * 1)) \Rightarrow (1, (\lambda \times 1)(1 * l), (1 \times \tilde{\pi} \cdot \kappa^{-1} \cdot (1 * l))(\tilde{\pi} \cdot \kappa \cdot \pi)),$$

consisting of, for each two spans, an equation of maps of spans,

- a trimodification:

$$l^*: \lambda \times 1 \Rightarrow \lambda \alpha,$$

consisting of:

- for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc}
 & (1 \times A) \times B & \\
 \tilde{\pi}_A^1 \times 1_B \swarrow & \downarrow l_{AB}^* & \searrow 1 \\
 (A \times B) & & (1 \times A) \times B \\
 \tilde{\pi}_{A \times B}^1 \pi_{1 \times (A \times B)}^{(1 \times A) \times B} \swarrow & & \searrow \pi_{(1 \times A) \times B}^{1 \times (A \times B)} \\
 & (1 \times (A \times B))((1 \times A) \times B) &
 \end{array}$$

$\pi_{(1 \times A) \times B}^{1 \times (A \times B)}$  (between  $(1 \times A) \times B$  and  $(1 \times (A \times B))((1 \times A) \times B)$ )  
 $\tilde{\pi}_{A \times B}^1 \pi_{1 \times (A \times B)}^{(1 \times A) \times B}$  (between  $(A \times B)$  and  $(1 \times (A \times B))((1 \times A) \times B)$ )

where  $l_{AB}^*$  is the unique 1-cell in  $\mathcal{B}$  such that:

$$\pi_{(1 \times A) \times B}^{1 \times (A \times B)} \cdot l_{AB}^* = 1, \quad \pi_{1 \times (A \times B)}^{(1 \times A) \times B} \cdot l_{AB}^* = a, \quad \text{and} \quad \kappa_{1 \times (A \times B)}^{1, a} \cdot l_{AB}^* = 1,$$

- and, for each two pairs  $(A, B), (A', B')$  of objects in  $\text{Span}(\mathcal{B})$ , an identity modification:

$$m_{l^*}: (1, (l^* * 1)(\lambda \times 1), \tilde{\pi}_{A' \times B'}^1 \cdot \kappa_{(A' \times B') \times 1}^{-1, (p' \times q') \times 1}) \Rightarrow$$

$$(1, (1 * \lambda)(\alpha * 1)(1 * l^*), (\tilde{\pi}_{A' \times B'}^1 \cdot \kappa_{(A' \times B') \times 1}^{-1, (p' \times q') \times 1}) \cdot \pi(\alpha * 1)(1 * l^*)),$$

consisting of, for each two spans, an equation of maps of spans,

- an identity counit perturbation,
- and, a unit perturbation:

$$\eta_l: (1, 1, 1) \Rightarrow (1, l^* l, 1),$$

consisting of, for each pair of objects  $A, B \in \text{Span}(\mathcal{B})$ , an isomorphism of maps of spans:

$$\eta_{l_{AB}}: (1, 1, 1) \Rightarrow (1, l_{AB}^* l_{AB}, 1),$$

where  $\eta_{l_{AB}}$  is the unique 2-cell in  $\mathcal{B}$  such that:

$$\pi_{(1 \times A) \times B}^{1 \times (A \times B)} \cdot \eta_{l_{AB}} = 1 \quad \text{and} \quad \pi_{1 \times (A \times B)}^{(1 \times A) \times B} \cdot \eta_{l_{AB}} = \kappa_{1 \times (A \times B)}^{-1, a}.$$

*Proof.* The modification axioms hold since  $m_l$ ,  $m_{l^*}$  and the modification components of  $\alpha$  and  $\lambda$  are all identities. The adjoint equivalence axioms follow.  $\square$

## Monoidal Middle Mediator Unit Modification

We define an adjoint equivalence:

$$m: (1 \times \lambda)\alpha \Rightarrow \rho \times 1$$

in the strict 2-category of tritransformations, trimodifications, and perturbations.

**Proposition 49.** *There is an adjoint equivalence  $(m, m^*, 1, \eta_m)$ , consisting of:*

- a trimodification:

$$m: (1 \times \lambda)\alpha \Rightarrow \rho \times 1,$$

consisting of:

- for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccccc}
 & (A \times (1 \times B))((A \times 1) \times B) & & & \\
 & \downarrow & & \searrow \pi_{(A \times 1) \times B}^{A \times (1 \times B)} & \\
 (1_A \times \lambda_B) \pi_{A \times (1 \times B)}^{(A \times 1) \times B} & & & & \\
 & \swarrow (1 \times \tilde{\pi}) \cdot \kappa^{-1} & & \searrow m_{AB} & \\
 (A \times B) & \xLeftrightarrow{\quad} & (A \times 1) \times B & & \\
 & \swarrow \rho_A \times 1_B & & \searrow 1 & \\
 & (A \times 1) \times B & & & 
 \end{array}$$

where  $(1 \times \tilde{\pi}) \cdot \kappa^{-1} := (1_A \times \tilde{\pi}_B^1) \cdot \kappa_{A \times (1 \times B)}^{-1, a}$  and  $m_{AB} := \pi_{(A \times 1) \times B}^{A \times (1 \times B)}$ ,

- and, for each two pairs  $(A, B), (A', B')$  of objects in  $\text{Span}(\mathcal{B})$ , an identity modification:

$$\begin{aligned}
 m_m: (1, (m * 1)(1 * (1 \times \lambda))(\alpha * 1), ((1 \times \lambda) \cdot \kappa^{-1}) \cdot \pi(\alpha * 1)) \Rightarrow \\
 (1, (\rho \times 1)(1 * m), ((\tilde{\pi} \cdot \kappa^{-1} \times 1) \cdot \pi(1 * m))((1 \times \tilde{\pi}) \cdot \kappa^{-1} \cdot \pi)),
 \end{aligned}$$

consisting of, for each two spans, an equation of maps of spans,

- a trimodification:

$$m^*: (\rho \times 1) \Rightarrow (1 \times \lambda)\alpha,$$

consisting of:

- for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccccc}
 & (A \times 1) \times B & & & \\
 & \downarrow & & \searrow 1 & \\
 \tilde{\pi}_A^1 \times 1_B & & & & \\
 & \swarrow & & \searrow m_{AB}^* & \\
 (A \times B) & \xLeftrightarrow{\quad} & (A \times 1) \times B & & \\
 & \swarrow (1_A \times \tilde{\pi}_B^1) \pi_{A \times (1 \times B)}^{(A \times 1) \times B} & & \searrow \pi_{(A \times 1) \times B}^{A \times (1 \times B)} & \\
 & (A \times (1 \times B))((A \times 1) \times B) & & & 
 \end{array}$$

where  $m_{AB}^*$  is the unique 1-cell such that

$$\pi_{(A \times 1) \times B}^{A \times (1 \times B)} \cdot m^* = 1 \quad \text{and} \quad \pi_{A \times (1 \times B)}^{(A \times 1) \times B} \cdot m^* = a,$$

and

$$\kappa_{A \times (1 \times B)}^{1, a} \cdot m^* = 1,$$



- and, for each two pairs  $(A, B), (A', B')$  of objects in  $\text{Span}(\mathcal{B})$ , an identity modification:

$$m_{m^*} : (1, (m^* * 1)(\rho \times 1), (\tilde{\pi}_B^1 \cdot \kappa^{-1} \times 1) \cdot \pi) \Rightarrow (1, (1 * (1 \times \lambda))(\alpha * 1)(1 * m^*), (1 \times \tilde{\pi} \cdot \kappa^{-1}) \cdot \pi(\alpha * 1)(1 * m^*)),$$

consisting of, for each two spans, an equation of maps of spans,

- an identity counit perturbation,

- and, a unit perturbation:

$$\eta_m : (1, 1, 1) \Rightarrow (1, m^* m, 1),$$

consisting of, for each pair of objects  $A, B \in \text{Span}(\mathcal{B})$ , an isomorphism of maps of spans:

$$\eta_{m_{AB}} : (1, 1, 1) \Rightarrow (1, m_{AB}^* m_{AB}, 1),$$

where  $\eta_{m_{AB}}$  is the unique 2-cell in  $\mathcal{B}$  such that:

$$\pi_{(A \times 1) \times B}^{A \times (1 \times B)} \cdot \eta_{m_{AB}} = 1 \quad \text{and} \quad \pi_{A \times (1 \times B)}^{(A \times 1) \times B} \cdot \eta_{m_{AB}} = \kappa^{-1 1, a}_{A \times (1 \times B)}.$$

*Proof.* The modification axioms hold since  $m_m$ ,  $m_{m^*}$  and the modification components of  $\alpha$ ,  $\lambda$ , and  $\rho$  are all identities. The adjoint equivalence axioms follow.  $\square$

### Monoidal Right Mediator Unit Modification

We define an adjoint equivalence:

$$r : (1 \times \rho)\alpha \Rrightarrow \rho$$

in the 2-category of transformations, modifications, and perturbations.

**Proposition 50.** *There is an adjoint equivalence  $(r, r^*, 1, \eta_r)$ , consisting of:*

- a trimodification:

$$r : (1 \times \rho)\alpha \Rrightarrow \rho,$$

consisting of:

- for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccc} & (A \times (B \times 1))((A \times B) \times 1) & \\ (1 \times \rho_B) \pi_{A \times (B \times 1)}^{A \times (B \times 1) \times 1} \swarrow & \downarrow r_{AB} & \searrow \pi_{(A \times B) \times 1}^{A \times (B \times 1)} \\ A \times B & \Downarrow & (A \times B) \times 1 \\ \rho_A \times 1_B \swarrow & \downarrow 1 & \nearrow \\ & (A \times B) \times 1 & \end{array}$$

where  $(1 \times \rho) \cdot \kappa^{-1} := (\tilde{\pi}_A^1 \times 1_B) \cdot \kappa^{-1 1, a}_{A \times (B \times 1)}$  and  $r_{AB} := \pi_{(A \times B) \times 1}^{A \times (B \times 1)}$ ,

- and, for each two pairs  $(A, B), (A', B')$  of objects in  $\text{Span}(\mathcal{B})$ , an identity modification:

$$m_r : (1, (r * 1)(1 * (1 \times \rho))(\alpha * 1), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi(\alpha * 1)) \Rightarrow (1, \rho(1 * r), \rho \cdot \kappa^{-1} \cdot \pi),$$

consisting of, for each two spans, an equation of maps of spans,

- a trimodification:

$$r^* : \rho \Rrightarrow (1 \times \rho)\alpha,$$

consisting of:

– for each pair of objects  $A, B$  in  $\text{Span}(\mathcal{B})$ , a map of spans:

$$\begin{array}{ccccc}
 & & (A \times B) \times 1 & & \\
 & \swarrow \rho_{A \times B} & \downarrow & \searrow 1 & \\
 A \times B & \cong & & \cong & (A \times B) \times 1 \\
 & \nwarrow (1_A \times \rho_B) \pi_{A \times (B \times 1)}^{(A \times B) \times 1} & \downarrow r_{AB}^* & \nearrow \pi_{(A \times B) \times 1}^{A \times (B \times 1)} & \\
 & & (A \times (B \times 1))((A \times B) \times 1) & & 
 \end{array}$$

where  $r_{AB}^*$  is the unique 1-cell in  $\mathcal{B}$  such that

$$\pi_{(A \times B) \times 1}^{A \times (B \times 1)} r_{AB}^* = 1 \quad \pi_{A \times (B \times 1)}^{(A \times B) \times 1} r_{AB}^* = a, \quad \text{and} \quad \kappa_{A \times (B \times 1)}^{1, a} \cdot r_{AB}^* = 1,$$

– and, for each two pairs  $(A, B), (A', B')$  of objects in  $\text{Span}(\mathcal{B})$ , an identity modification:

$$m_{r^*}: (1, (r^* * 1)(\rho * 1), 1) \Rightarrow (1, (1 * (1 \times \rho))(\alpha * 1)(1 * r^*), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi(\alpha * 1)(1 * r^*)),$$

consisting of, for each two spans, an equation of maps of spans,

- an identity counit perturbation,
- and, a unit perturbation:

$$\eta_r: (1, 1, 1) \Rightarrow (1, r^* r, 1),$$

consisting of, for each pair of objects  $A, B \in \text{Span}(\mathcal{B})$ , an isomorphism of maps of spans:

$$\eta_{r_{AB}}: (1, 1, 1) \Rightarrow (1, r_{AB}^* r_{AB}, 1)$$

where  $\eta_{r_{AB}}$  is the unique 2-cell in  $\mathcal{B}$  such that:

$$\pi_{(A \times B) \times 1}^{A \times (B \times 1)} \cdot \eta_{r_{AB}} = 1 \quad \text{and} \quad \pi_{A \times (B \times 1)}^{(A \times B) \times 1} \cdot \eta_{r_{AB}} = \kappa_{A \times (B \times 1)}^{-1, a}.$$

*Proof.* The modification axioms hold since  $m_r, m_{r^*}$  and the modification components of  $\alpha$  and  $\rho$  are all identities. The adjoint equivalence axioms follow.  $\square$

*Proof.* The equations of 1-cells and 2-cells for  $m_r$  are straightforward to verify by uniqueness and definitions. The modification axioms hold since  $m_r$  and the modification components of  $\alpha$  and  $\rho$  are all identities.  $\square$

## 5.7 Monoidal Perturbations

### $K^5$ Perturbation

**Proposition 51.** *There is a perturbation consisting of, for each 5-tuple  $A, B, C, D, E$  of objects in  $\text{Span}(\mathcal{B})$ , an isomorphism of maps of spans:*

$$K_{ABCDE}^5: (1, (\Pi * 1)(1 * 1 * \alpha_{\mu^{-1}, 1, \alpha})(1 * \Pi * 1)((\Pi \times 1) * 1 * 1 * 1), 1) \Rightarrow$$

$$((a^{-1} \cdot \kappa) \cdot (1 * \Pi * 1)(1 * 1 * 1 * (1 \times \Pi))(1 * 1 * \alpha_{\mu^{-1}, \alpha, 1} * 1 * 1), (1 * \Pi)(\alpha_{\mu^{-1}, \alpha, 1} * 1 * 1)(1 * \Pi * 1)(1 * 1 * 1 * (1 \times \Pi))(1 * 1 * \alpha_{\mu^{-1}, \alpha, 1} * 1 * 1), 1),$$

defined by the unique 2-cell:

$$\begin{aligned}
 K_{ABCDE}^5: & (\Pi * 1)(1 * 1 * \alpha_{\mu^{-1}, 1, \alpha})(1 * \Pi * 1)((\Pi \times 1) * 1 * 1 * 1) \Rightarrow \\
 & (1 * \Pi)(\alpha_{\mu^{-1}, \alpha, 1} * 1 * 1)(1 * \Pi * 1)(1 * 1 * 1 * (1 \times \Pi))(1 * 1 * \alpha_{\mu^{-1}, \alpha, 1} * 1 * 1),
 \end{aligned}$$

in  $\mathcal{B}$  such that:

$$\pi_{((AB)C)D)E}^{((AB)C)(DE)} \pi_{((AB)C)(DE))(((AB)C)D)E} \cdot K^5 = \kappa_{((AB)C)D)E}^{1, (a \times 1) \times 1},$$

$$\pi_{((AB)C)(DE))E}^{((AB)C)D)E} \pi_{((AB)C)(DE))(((AB)C)D)E} \cdot K^5 = \kappa_{((AB)C)D)E}^{-1, a \times 1} \circ \kappa_{A((BC)D))E}^{-1, 1, (1 \times a) \times 1} \circ \kappa_{A(B(CD))E}^{-1, 1, a} \circ \kappa_{A((BC)D))E}^{-1, 1 \times a},$$

and

$$\pi_{(AB)(C(DE))}^{((AB)C)(DE))(((AB)C)D)E} \cdot K^5 = 1.$$

*Proof.*  $\square$

### $U_{4,1}$ Perturbation

**Proposition 52.** *There is a perturbation consisting of, for each triple  $A, B, C$  of objects in  $\text{Span}(\mathcal{B})$ , an isomorphism of maps of spans:*

$$U_{ABC}^{4,1}: (a^{-1} \cdot \kappa \cdot (1 * l)(\Pi * 1), \alpha_{\mu \cdot \lambda, 1, 1}(1 * l)(\Pi * 1), \kappa^{-1} \cdot \pi(\Pi * 1)) \Rightarrow \\ (1, ((l \times 1) * 1)(1 * l * 1)(1 * 1 * \lambda_\alpha), \tilde{\pi} \cdot \kappa^{-1} \cdot \pi)$$

defined by the unique 2-cell:

$$U_{ABC}^{4,1}: \alpha_{\mu \cdot \lambda, 1, 1}(1 * l)(\Pi * 1) \Rightarrow ((l \times 1) * 1)(1 * l * 1)(1 * 1 * \lambda_\alpha)$$

in  $\mathcal{B}$  such that:

$$\pi_{((1A)B)C}^{(AB)C} \cdot U_{4,1} = \kappa_{((1A)B)C}^{-1, a \times 1} \circ \kappa_{(1(AB))C}^{-1, a}$$

and

$$\pi_{(AB)C}^{((1A)B)C} \cdot U_{4,1} = \kappa_{1((AB)C)}^{1, a \times 1}.$$

*Proof.* The equation of 2-cells in  $\mathcal{B}$ :

$$(1 \cdot (\pi_{(AB)C}^{((1A)B)C} \cdot U_{4,1}))(\kappa_{(AB)C}^{1, (\lambda_A \times 1) \times 1} \cdot (\alpha_{\lambda, 1, 1}^{-1}(1 * l)(\Pi * 1))) = \\ (\kappa_{(AB)C}^{1, (\lambda_A \times 1) \times 1} \cdot (((l \times 1) * 1)(1 * l * 1)(1 * 1 * \lambda_\alpha))((\lambda_A \times 1_B) \times 1_C)) \cdot (\pi_{((1A)B)C}^{(AB)C} \cdot U_{4,1}))$$

allows us to apply the universal property to obtain the the 2-cell  $U_{4,1}$ . The perturbation axiom is satisfied since the components of the relevant modifications are identities.  $\square$

### $U_{4,2}$ Perturbation

**Proposition 53.** *There is a perturbation consisting of, for each triple  $A, B, C$  of objects in  $\text{Span}(\mathcal{B})$ , an isomorphism of maps of spans:*

$$U_{ABC}^{4,2}: (a^{-1} \cdot \kappa \cdot (1 * m)(\Pi * 1), \alpha_{\mu \cdot \rho, 1, 1}(1 * m)(\Pi * 1), (1 \times \tilde{\pi}) \cdot \kappa^{-1} \cdot \pi(\Pi * 1)) \Rightarrow \\ (1, ((m \times 1) * 1)(1 * \alpha_{1, \lambda, 1}^{-1})(1 * 1 * (1 \times l)), (1 \times \kappa^{-1}) \cdot \pi)$$

defined by the unique 2-cell:

$$U_{ABC}^{4,2}: \alpha_{\rho, 1, 1}^{-1}(1 * m)(\Pi * 1) \Rightarrow ((m \times 1) * 1)(1 * \alpha_{1, \lambda, 1}^{-1})(1 * 1 * (1 \times l))$$

in  $\mathcal{B}$  such that:

$$\pi_{((A1)B)C}^{(AB)C} \cdot U_{4,2} = \kappa_{((A1)B)C}^{-1, a \times 1} \circ \kappa_{(A(1B))C}^{-1, a}$$

and

$$\pi_{(AB)C}^{((AB)1)C} \cdot U_{4,2} = 1.$$

*Proof.* The equation of 2-cells in  $\mathcal{B}$ :

$$(1 \cdot (\pi_{(AB)C}^{((1A)B)C} \cdot U_{4,2}))(\kappa_{(AB)C}^{1, (\lambda_A \times 1) \times 1} \cdot (\alpha_{\rho, 1, 1}(1 * m)(1 * \Pi))) = \\ (\kappa_{(AB)C}^{1, (\lambda_A \times 1) \times 1} \cdot (((m \times 1) * 1)(1 * \alpha_{1, \lambda, 1}^{-1})(1 * 1 * (1 \times l))((\lambda_A \times 1_B) \times 1_C)) \cdot (\pi_{((1A)B)C}^{(AB)C} \cdot U_{4,2}))$$

allows us to apply the universal property to obtain the the 2-cell  $U_{4,2}$ . The perturbation axiom is satisfied since the components of the relevant modifications are identities.  $\square$

### $U_{4,3}$ Perturbation

**Proposition 54.** *There is a perturbation consisting of, for each triple  $A, B, C$  of objects in  $\text{Span}(\mathcal{B})$ , an isomorphism of maps of spans:*

$$U_{ABC}^{4,3}: (a^{-1} \cdot \kappa \cdot (\Pi * 1), (m * 1)(1 * \alpha_{\mu^{-1}(1,1,\rho)})(\Pi * 1), 1) \Rightarrow \\ (1, ((l \times 1) * 1)(1 * \alpha_{\mu^{-1}, \lambda, 1})(1 * (1 * (1 \times m))), (1 \times ((1 \times \tilde{\pi}) \cdot \kappa^{-1})) \cdot \pi)$$

defined by the unique 2-cell:

$$U_{ABC}^{4,3}: (m * 1)(1 * \alpha_{\mu^{-1}(1,1,\rho)})(\Pi * 1) \Rightarrow ((l \times 1) * 1)(1 * \alpha_{\mu^{-1}, \lambda, 1})(1 * (1 * (1 \times m)))$$

in  $\mathcal{B}$  such that:

$$\pi_{((AB)1)C}^{(AB)C} \cdot U_{4,3} = 1$$

and

$$\pi_{(AB)C}^{((AB)1)C} \cdot U_{4,3} = \kappa_{A((B1)C)}^{-1, 1 \times \alpha}.$$

*Proof.* The equation of 2-cells in  $\mathcal{B}$ :

$$(1 \cdot (\pi_{(AB)C}^{((AB)1)C} \cdot U_{4,3}))(\kappa_{(AB)C}^{1, \rho_{AB} \times 1} \cdot (m * 1)(1 * \alpha_{1,1,\lambda}^{-1})(\Pi * 1)) = \\ (\kappa_{(AB)C}^{1, \rho_{AB} \times 1} \cdot (((r \times 1) * 1)(1 * \alpha_{1,\lambda,1}^{-1})(1 * 1 * (1 \times m))))((\rho_{AB} \times 1_C) \cdot (\pi_{(AB)C}^{((AB)1)C} \cdot U_{4,3})).$$

allows us to apply the universal property to obtain the the 2-cell  $U_{4,3}$ . The perturbation axiom is satisfied since the components of the relevant modifications are identities.  $\square$

### $U_{4,4}$ Perturbation

**Proposition 55.** *There is a perturbation consisting of, for each triple  $A, B, C$  of objects in  $\text{Span}(\mathcal{B})$ , an isomorphism of maps of spans:*

$$U_{ABC}^{4,4}: (1, (r * 1)(1 * \alpha_{\mu^{-1}, 1, \rho})(\Pi * 1), 1) \Rightarrow \\ (1, \rho_\alpha(1 * \rho)(1 * 1 * (1 \times r)), (\tilde{\pi} \cdot \kappa^{-1} \cdot (1 * \rho)(1 * 1 * (1 \times r)))(\tilde{\pi} \cdot \kappa^{-1} \cdot \pi(1 * 1 * (1 \times r))(((1 \times ((1 \times \rho) \cdot \kappa^{-1})) \cdot \pi)))$$

defined by the unique 2-cell:

$$U_{ABC}^{4,4}: \lambda(1 * l)(1 * 1 * (1 \times l))((1 \times 1) \times \lambda_{(AB)C}) \Rightarrow \lambda^{-1}(1 * l)(1 * 1 * (1 \times l))$$

in  $\mathcal{B}$  such that:

$$\pi_{((AB)C)1}^{(AB)C} \cdot U_{4,4} = 1$$

and

$$\pi_{(AB)C}^{((AB)C)1} \cdot U_{4,4} = \kappa_{A((B1)C)}^{-1, 1 \times a}.$$

*Proof.* The equation of 2-cells in  $\mathcal{B}$ :

$$(1 \cdot (\pi_{(AB)C}^{((AB)C)1} \cdot U_{4,4}))(\kappa_{(AB)C}^{1, \rho_{(AB)C}} \cdot (r * 1)(1 * \alpha_{1,1,\rho}^{-1})(\Pi * 1)) = \\ ((\kappa_{(AB)C}^{1, \rho_{(AB)C}} \cdot \rho_\alpha(1 * \rho)(1 * 1 * (1 \times r))((1 \times 1) \times \lambda_{(AB)C}) \cdot (\pi_{((AB)C)1}^{(AB)C} \cdot U_{4,4}))$$

allows us to apply the universal property to obtain the the 2-cell  $U_{4,4}$ . The perturbation axiom is satisfied since the components of the relevant modifications are identities.  $\square$

## 5.8 Monoidal Tricategory Axioms

There are four axioms that a monoidal tricategory must satisfy. These were written down by Trimble in diagram form over the span of forty-six pages [24]. These diagrams are reproduced in Definition 6 along with clarifications and explanations of the geometric 2- and 3-cells comprising the diagrams. Checking the axioms is not terribly difficult in our construction; however, certain “twistings” sometimes make opaque the constituent cells of the geometric 3-cell domains and codomains.

## The $K_6$ Associativity Axiom

**Proposition 56.** *The  $K_6$  axiom for the monoidal structure on  $\text{Span}(\mathcal{B})$  holds.*

*Proof.* We have:

$$\pi_{(((AB)C)D)E)F} \cdot (1 * K_5)(1 * \alpha * 1)(1 * K_5 * 1)(1 * \Pi * 1)(1 * \alpha * 1)(1 * K_5) = \\ 1 \circ 1 \circ \kappa_{((A((BC)D))E)F}^{1,((1 \times a) \times 1) \times 1} \circ 1 \circ 1 \circ \left( \kappa_{((A((BC)D))E)F}^{1,(a \times 1) \times 1} \circ \kappa_{(((AB)C)D)E)F}^{1,((a \times 1) \times 1) \times 1} \right)$$

and

$$\pi_{(((AB)C)D)E)F} \cdot (1 * \Pi * 1)(1 * (1 \times K_5 * 1)(1 * \alpha * 1)(1 * K_5 * 1)(1 * \Pi * 1)(1 * \Pi * 1)(1 * K_5 * 1)((K_5 \times 1) * 1) = \\ 1 \circ 1 \circ 1 \circ \left( \kappa_{((A((BC)D))E)F}^{1,((1 \times a) \times 1) \times 1} \circ \kappa_{((A((BC)D))E)F}^{1,(a \times 1) \times 1} \right) \circ 1 \circ 1 \circ 1 \circ \kappa_{(((AB)C)D)E)F}^{1,((a \times 1) \times 1) \times 1}$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

We have:

$$\pi_{((AB)C)D)E)F} \cdot (1 * K_5)(1 * \alpha * 1)(1 * K_5 * 1)(1 * \Pi * 1)(1 * \alpha * 1)(1 * K_5) = \\ \circ \kappa_{(A(B((CD)E)))F}^{-1,1 \times (1 \times a) \times 1} \circ \kappa_{(A(B(C(DE))))F}^{-1,1,a} \circ \kappa_{A(B(C(DE)))F}^{-1,1 \times (a \times 1)} \circ \kappa_{A(B((C(DE))F))}^{-1,1 \times (1 \times a)} \circ 1 \circ 1 \\ 1 \circ \kappa_{((A((BC)D))E)F}^{1,((1 \times a) \times 1) \times 1} \circ 1 \circ \left( \circ \kappa_{((A((BC)D))E)F}^{-1,1,((1 \times a) \times 1) \times 1} \circ \kappa_{((A(B(CD))E)F}^{-1,1,a \times 1} \circ \kappa_{(A(B((CD)E))F}^{-1,1,a \times 1} \right)$$

and

$$\pi_{((AB)C)D)E)F} \cdot (1 * \Pi * 1)(1 * (1 \times K_5 * 1)(1 * \alpha * 1)(1 * K_5 * 1)(1 * \Pi * 1)(1 * \Pi * 1)(1 * K_5 * 1)((K_5 \times 1) * 1) = \\ \circ \kappa_{(A(B(C(DE))))F}^{-1,1,a} \circ \kappa_{A(B(C(DE)))F}^{-1,1 \times (a \times 1)} \circ \kappa_{A(B((C(DE))F))}^{-1,1 \times (1 \times a)} \circ 1 \circ 1 \circ 1 \circ 1 \\ 1 \circ 1 \circ 1 \circ \left( \kappa_{((A(B(CD))E)F}^{-1,1,a \times 1} \circ \kappa_{(A(B((CD)E))F}^{-1,1,a \times 1} \circ \kappa_{(A(B((CD)E))F}^{-1,1,(1 \times (1 \times a)) \times 1} \right)$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

We have:

$$\pi_{((AB)C)D)E)F} \cdot (1 * K_5)(1 * \alpha * 1)(1 * K_5 * 1)(1 * \Pi * 1)(1 * \alpha * 1)(1 * K_5) = \\ \circ \kappa_{(A(B(C(DE))))F}^{-1,1,a} \circ \kappa_{A(B(C(DE)))F}^{-1,1 \times (a \times 1)} \circ \kappa_{A(B((C(DE))F))}^{-1,1 \times (1 \times a)} \circ 1 \circ 1 \circ 1 \circ 1 \circ 1 \\ \left( \kappa_{((A(B(CD))E)F}^{-1,1,a \times 1} \circ \kappa_{(A(B((CD)E))F}^{-1,1,a \times 1} \circ \kappa_{(A(B((CD)E))F}^{-1,1,(1 \times (1 \times a)) \times 1} \right)$$

and

$$\pi_{((AB)C)D)E)F} \cdot (1 * \Pi * 1)(1 * (1 \times K_5 * 1)(1 * \alpha * 1)(1 * K_5 * 1)(1 * \Pi * 1)(1 * \Pi * 1)(1 * K_5 * 1)((K_5 \times 1) * 1) = \\ \circ \kappa_{(A(B(C(DE))))F}^{-1,1,a} \circ \kappa_{A(B(C(DE)))F}^{-1,1 \times (a \times 1)} \circ \kappa_{A(B((C(DE))F))}^{-1,1 \times (1 \times a)} \circ 1 \circ 1 \circ 1 \circ 1 \circ 1 \\ 1 \circ 1 \circ 1 \circ \left( \kappa_{((A(B(CD))E)F}^{-1,1,a \times 1} \circ \kappa_{(A(B((CD)E))F}^{-1,1,(1 \times a) \times 1} \circ \kappa_{(A(B((CD)E))F}^{-1,1,(1 \times (1 \times a)) \times 1} \right)$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

We have:

$$\pi_{(AB)C)D)E)F} \cdot (1 * K_5)(1 * \alpha * 1)(1 * K_5 * 1)(1 * \Pi * 1)(1 * \alpha * 1)(1 * K_5) = \\ 1 \circ 1 \circ 1 \circ 1 \circ 1 \circ 1 \circ 1$$

and

$$\pi_{(AB)(C(D(EF)))} \cdot (1 * \Pi * 1)(1 * (1 \times K_5 * 1)(1 * \alpha * 1)(1 * K_5 * 1)(1 * \Pi * 1)(1 * \Pi * 1)(1 * K_5 * 1)((K_5 \times 1) * 1) =$$

$$1 \circ 1 \circ 1 \circ 1 \circ 1 \circ 1 \circ 1 \circ 1$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

The axiom then holds by the universal property.  $\square$

### The $U_{5,2}$ Unit Axiom

**Proposition 57.** *The  $U_{5,2}$  axiom for the monoidal structure on  $\text{Span}(\mathcal{B})$  holds.*

*Proof.* It is straightforward to verify the axiom. We have:

$$\pi_{(((A1)B)C)D} \cdot (1 * (1 \times U_{4,1}))(1 * \Pi * 1)(1 * \alpha)(1 * U_{4,2} * 1)(K_5 * 1) =$$

$$1 \circ 1 \circ \kappa_{(((A1)B)C)D}^{-1, (a \times 1) \times 1} \circ \left( \kappa_{((A(1B))C)D}^{-1, a \times 1} \circ \kappa_{(A((1B)C))D}^{-1, a} \circ \kappa_{A(((1B)C)D)}^{-1, (1 \times a) \times 1} \circ \kappa_{A((1(BC))D)}^{-1, 1 \times a} \right) \circ 1$$

and

$$\pi_{(((A1)B)C)D} \cdot (1 * \alpha * 1)(1 * (U_{4,1} \times 1) * 1)(1 * U_{4,2} * 1)(1 * m * 1)(1 * \Pi) =$$

$$1 \circ \left( \kappa_{(((A1)B)C)D}^{-1, (a \times 1) \times 1} \circ \kappa_{((A(1B))C)D}^{-1, a \times 1} \right) \circ \left( \kappa_{(A((1B)C))D}^{-1, a} \circ \kappa_{A(((1B)C)D)}^{-1, (1 \times a) \times 1} \right) \circ \kappa_{A((1(BC))D)}^{-1, 1 \times a} \circ 1$$

up to whiskering by structural 1-cells.

We have:

$$\pi_{((AB)C)D} \cdot (1 * (1 \times U_{4,1}))(1 * \Pi * 1)(1 * \alpha)(1 * U_{4,2} * 1)(K_5 * 1) =$$

$$\circ \left( \kappa_{((A(1B))C)D}^{-1, a \times 1} \circ \kappa_{(A((1B)C))D}^{-1, a} \circ \kappa_{A(((1B)C)D)}^{-1, (1 \times a) \times 1} \circ \kappa_{A((1(BC))D)}^{-1, 1 \times a} \right) \circ 1$$

$$\left( \kappa_{A(((1B)C)D)}^{1, (1 \times a) \times 1} \circ \kappa_{A((1(BC))D)}^{1, 1 \times a} \right)^{-1} \circ 1 \circ \left( \kappa_{A((1(BC))D)}^{1, 1 \times a} \circ \kappa_{A(((1B)C)D)}^{1, (1 \times a) \times 1} \circ \kappa_{(A((1B)C))D}^{1, a} \circ \kappa_{((A(1B))C)D}^{1, a \times 1} \right)$$

$$\left( \kappa_{A(((1B)C)D)}^{1, (1 \times a) \times 1} \circ \kappa_{A((1(BC))D)}^{1, 1 \times a} \right)^{-1} \circ 1 \circ 1 =$$

and

$$\pi_{((AB)C)D} \cdot (1 * \alpha * 1)(1 * (U_{4,1} \times 1) * 1)(1 * U_{4,2} * 1)(1 * m * 1)(1 * \Pi) =$$

$$1 \circ 1 \circ \kappa_{A(((1B)C)D)}^{-1, (1 \times a) \times 1} \circ \kappa_{A((1(BC))D)}^{-1, 1 \times a} \circ 1$$

up to whiskering by structural 1-cells.

We have:

$$\pi_{(AB)(CD)} \cdot (1 * (1 \times U_{4,1}))(1 * \Pi * 1)(1 * \alpha)(1 * U_{4,2} * 1)(K_5 * 1) =$$

$$1 \circ 1 \circ 1 \circ 1 \circ 1$$

and

$$\pi_{(AB)(CD)} \cdot (1 * \alpha * 1)(1 * (U_{4,1} \times 1) * 1)(1 * U_{4,2} * 1)(1 * m * 1)(1 * \Pi) =$$

$$1 \circ 1 \circ 1 \circ 1 \circ 1$$

up to whiskering by structural 1-cells.

The axiom then holds by the universal property.  $\square$

**Proposition 58.** *The  $U_{5,3}$  axiom for the monoidal structure on  $\text{Span}(\mathcal{B})$  holds.*

*Proof.* We have:

$$\pi_{(((AB)1)C)D} \cdot (1 * \alpha * 1)(1 * (U_{4,3} \times 1) * 1)(1 * (1 \times \Pi) * 1)(1 * \alpha * 1)(1 * (1 \times U_{4,2})) =$$

$$1 \circ 1 \circ 1 \circ 1 \circ \left( \kappa^{-1^{1, (a \times 1) \times 1}}_{(((AB)1)C)D} \circ \kappa^{-1^{1, a \times 1}}_{((A(B1))C)D} \circ \kappa^{-1^{1, a}}_{(A((B1)C))D} \circ \kappa^{-1^{1, (1 \times a) \times 1}}_{A(((B1)C)D)} \circ \kappa^{-1^{1, 1 \times a}}_{A((B(1C))D)} \right)$$

and

$$\pi_{(((AB)1)C)D} \cdot ((1 \times U_{4,2}) * 1)(1 * \Pi)(1 * \alpha)(1 * U_{4,3} * 1)(K_5 * 1) =$$

$$1 \circ 1 \circ \kappa^{-1^{1, (a \times 1) \times 1}}_{(((AB)1)C)D} \circ 1 \circ \left( \kappa^{-1^{1, a \times 1}}_{((A(B1))C)D} \circ \kappa^{-1^{1, a}}_{(A((B1)C))D} \circ \kappa^{-1^{1, (1 \times a) \times 1}}_{A(((B1)C)D)} \circ \kappa^{-1^{1, 1 \times a}}_{A((B(1C))D)} \right)$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

We have:

$$\pi_{((AB)C)D} \cdot (1 * \alpha * 1)(1 * (U_{4,3} \times 1) * 1)(1 * (1 \times \Pi) * 1)(1 * \alpha * 1)(1 * (1 \times U_{4,2})) =$$

$$1 \circ \left( \kappa^{-1^{1, (a \times 1) \times 1}}_{(((AB)1)C)D} \circ \kappa^{-1^{1, a \times 1}}_{((A(B1))C)D} \circ \kappa^{-1^{1, a}}_{(A((B1)C))D} \circ \kappa^{-1^{1, (1 \times a) \times 1}}_{A(((B1)C)D)} \right) \circ \kappa^{-1^{1, 1 \times a}}_{A((B(1C))D)} \circ 1 \circ 1$$

and

$$\pi_{((AB)C)D} \cdot ((1 \times U_{4,2}) * 1)(1 * \Pi)(1 * \alpha)(1 * U_{4,3} * 1)(K_5 * 1) =$$

$$1 \circ 1 \circ \kappa^{-1^{1, (a \times 1) \times 1}}_{(((AB)1)C)D} \circ 1 \circ \left( \kappa^{-1^{1, a \times 1}}_{((A(B1))C)D} \circ \kappa^{-1^{1, a}}_{(A((B1)C))D} \circ \kappa^{-1^{1, (1 \times a) \times 1}}_{A(((B1)C)D)} \circ \kappa^{-1^{1, 1 \times a}}_{A((B(1C))D)} \right)$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

We have:

$$\pi_{(AB)(CD)} \cdot (1 * \alpha * 1)(1 * (U_{4,3} \times 1) * 1)(1 * (1 \times \Pi) * 1)(1 * \alpha * 1)(1 * (1 \times U_{4,2})) =$$

$$1 \circ 1 \circ 1 \circ \kappa^{-1^{1, 1 \times (1 \times a)}}_{A(B((1C)D))} \circ 1$$

and

$$\pi_{(AB)(CD)} \cdot ((1 \times U_{4,2}) * 1)(1 * \Pi)(1 * \alpha)(1 * U_{4,3} * 1)(K_5 * 1) =$$

$$1 \circ 1 \circ 1 \circ \kappa^{-1^{1, 1 \times (1 \times a)}}_{A(B((1C)D))} \circ 1$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

The axiom then holds by the universal property. □

**Proposition 59.** *The  $U_{5,4}$  axiom for the monoidal structure on  $\text{Span}(\mathcal{B})$  holds.*

*Proof.* We have:

$$\pi_{(((AB)C)1)D} \cdot (1 * \alpha * 1)(1 * (U_{4,4} \times 1) * 1)(1 * \Pi * 1)(1 * \alpha * 1)(1 * U_{4,3}) = 1 \circ 1 \circ 1 \circ 1 \circ 1$$

and

$$\pi_{(((AB)C)1)D} \cdot ((1 \times U_{4,3}) * 1)(1 * U_{4,3} * 1)(1 * m * 1)(1 * \Pi * 1)(K_5 * 1) =$$

$$1 \circ 1 \circ \kappa^{-1^{1, (a \times 1) \times 1}}_{(((AB)C)1)D} \circ 1 \circ \kappa^{1, (a \times 1) \times 1}_{(((AB)C)1)D}$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

We have:

$$\pi_{((AB)C)D} \cdot (1 * \alpha * 1)(1 * (U_{4,4} \times 1) * 1)(1 * \Pi * 1)(1 * \alpha * 1)(1 * U_{4,3}) =$$

$$1 \circ \left( \kappa^{-1^{1, (a \times 1) \times 1}}_{(((AB)C)1)D} \circ \kappa^{-1^{1, a \times 1}}_{((A(BC))1)D} \circ \kappa^{-1^{1, a}}_{(A((BC)1))D} \circ \kappa^{-1^{1, (1 \times a) \times 1}}_{A(((BC)1)D)} \circ \kappa^{-1^{1, 1 \times a}}_{A((B(C1))D)} \right) \circ 1 \circ 1 \circ 1$$

and

$$\pi_{((AB)C)D} \cdot ((1 \times U_{4,3}) * 1)(1 * U_{4,3} * 1)(1 * m * 1)(1 * \Pi * 1)(K_5 * 1) = \\ 1 \circ 1 \circ \left( \kappa^{-1^{1, (a \times 1) \times 1}}_{(((AB)C)1)D} \circ \kappa^{-1^{1, a \times 1}}_{((A(BC))1)D} \circ \kappa^{-1^{1, a}}_{(A((BC)1))D} \circ \kappa^{-1^{1, (1 \times a) \times 1}}_{A(((BC)1)D)} \circ \kappa^{-1^{1, 1 \times a}}_{A((B(C1))D)} \right) \circ 1 \circ 1$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

We have:

$$\pi_{(AB)(CD)} \cdot (1 * \alpha * 1)(1 * (U_{4,4} \times 1) * 1)(1 * \Pi * 1)(1 * \alpha * 1)(1 * U_{4,3}) = \\ 1 \circ 1 \circ 1 \circ \kappa^{-1^{1, 1 \times (1 \times a)}}_{A(B((C1)D))} \circ 1$$

and

$$\pi_{(AB)(CD)} \cdot ((1 \times U_{4,3}) * 1)(1 * U_{4,3} * 1)(1 * m * 1)(1 * \Pi * 1)(K_5 * 1) = \\ \kappa^{-1^{1, 1 \times (1 \times a)}}_{A(B((C1)D))} \circ 1 \circ 1 \circ 1 \circ 1$$

up to whiskering by associator 1-cells  $a$  and projection 1-cells  $\tilde{\pi}_A^1$ .

The axiom then holds by the universal property.  $\square$

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## References

- [1] M. Aguiar and S. Mahajan, Monoidal functors, species and Hopf algebras. Available at .
- [2] J. Baez and J. Dolan, From finite sets to Feynman diagrams, in Mathematics Unlimited - 2001 and Beyond, eds. Bjorn Engquist and Wilfried Schmid, Springer, Berlin, 2001, pp. 29-50. Also available as [arXiv:math/0004133](https://arxiv.org/abs/math/0004133).
- [3] J. Baez, A. E. Hoffnung, and C. Walker, Higher Dimensional Algebra VII: Groupoidification, in Theory and Applications of Categories. Available at [arXiv:0908.4305](https://arxiv.org/abs/0908.4305).
- [4] J. Baez and A. E. Hoffnung, Higher Dimensional Algebra VIII: The Hecke Bicategory. Available at <http://math.ucr.edu/home/baez/hecke.pdf>.
- [5] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, Springer, Berlin, (1967), 1-77.
- [6] Bird, Kelly, Power, Street, Flexible Limits for 2-Categories, *J. Pure and Appl. Alg.*, **61**, 1989, pp. 1-27.
- [7] R. Blackwell, G. M. Kelly and A. J. Power, Two-dimensional monad theory, *Jour. Pure Appl. Algebra* **59** (1989), 1-41.
- [8] A. Carboni and R. F. C. Walters. Cartesian bicategories I. *J. Pure Appl. Algebra* **49** (1987), 11-32.
- [9] S. Carmodey. Ph.D. Thesis.
- [10] R. Gordon, J. Power, and R. Street. Coherence for tricategories. *Mem. Amer. Math. Soc.* **117** (1995).
- [11] N. Gurski, An algebraic theory of tricategories, PhD thesis, University of Chicago, June 2006.
- [12] A. Joyal and R. Street. Pullbacks equivalent to pseudopullbacks. *Cahiers Topologie Géom. Diff.* **34(2)** (1993), 153-156.
- [13] G. M. Kelly, Elementary observations on 2-categorical limits, *Bull. Austral. Math. Soc.* **39** (1989) 301-317.



- [14] G. M. Kelly, Basic Concepts of Enriched Category Theory, *London Mathematical Society Lecture Notes Series* **64** (Cambridge University Press, 1982).
- [15] G. M. Kelly and R. Street. Review of the elements of 2-categories. *Lecture Notes in Math.* **420** (1974), 75-103.
- [16] T. Kenney and D. Pronk, The span profunctor and generalised span constructions. Unpublished draft.
- [17] T. Leinster, Basic Bicategories. Available as [arXiv:math/9810017](https://arxiv.org/abs/math/9810017).
- [18] P. McCrudden. Balanced coalgebroids. *Theory Applic. Categ.* **7** (2000), 71-147.
- [19] A. J. Power, Coherence for bicategories with finite bilimits I, *Contemporary Math.* **92** (1989), 341–347.
- [20] J. D. Stasheff, Homotopy Associativity of  $H$ -Spaces I, *Trans. Am. Math. Soc.* **108**, Issue 2 (1963), 275-292.
- [21] J. D. Stasheff, Homotopy Associativity of  $H$ -Spaces II, *Trans. Am. Math. Soc.* **108**, Issue 2 (1963), 293-312.
- [22] R. Street, Fibrations in bicategories, *Cahiers Topol. Géom. Diff.* **21** (1980), 111-160.
- [23] R. Street, Limits indexed by category-valued 2-functors, *J. Pure Appl. Algebra* **8** (1976), 149-181.
- [24] T. Trimble, Notes on Tetracategories. Available as [math.ucr.edu/home/baez/trimble/tetracategories.html](http://math.ucr.edu/home/baez/trimble/tetracategories.html).
- [25] T. Trimble, Private communication.